K-THEORY OF ENDOMORPHISMS
VIA NONCOMMUTATIVE MOTIVES

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ABSTRACT. In this article we study the K-theory of endomorphisms using noncommutative motives. We start by extending the K-theory of endomorphisms functor from ordinary rings to (stable) ∞-categories. We then prove that this extended functor KEnd$(-)$ not only descends to the category of noncommutative motives but moreover becomes co-represented by the noncommutative motive associated to the tensor algebra $\mathbb{S}[t]$ of the sphere spectrum $\mathbb{S}$. Using this co-representability result, we then classify all the natural transformations of KEnd$(-)$ in terms of an integer plus a fraction between polynomials with constant term 1; this solves a problem raised by Almkvist in the seventies. Finally, making use of the multiplicative co-algebra structure of $\mathbb{S}[t]$, we explain how the (rational) Witt vectors can also be recovered from the symmetric monoidal category of noncommutative motives. Along the way we show that the K_0-theory of endomorphisms of a *connective* ring spectrum R equals the K_0-theory of endomorphisms of the underlying ordinary ring $\pi_0 R$.

CONTENTS

1. INTRODUCTION

K-theory of endomorphisms. The K-theory of endomorphisms was introduced in the seventies by Almkvist [1, 2] and Grayson [19, 20]. Given an ordinary ring A, let $\mathbf{P}(A)$ be the category of finitely generated projective (right) A-modules and End$(\mathbf{P}(A))$ the associated category of endomorphisms: its objects are the pairs (M, α), with $M \in \mathbf{P}(A)$ and α an endomorphism of M, and its morphisms $(M, \alpha) \to (M', \alpha')$ are the A-linear homomorphisms $f : M \to M'$ verifying the equality $f\alpha = \alpha' f$. Note that this latter category inherits naturally from $\mathbf{P}(A)$ an exact structure in the sense of Quillen [32]. The classical K-*theory of endomorphisms of* A was then defined as the homotopy groups of the (connective) algebraic K-theory spectrum KEnd$(\mathbf{P}(A))$ of the exact category End$(\mathbf{P}(A))$.

Bloch [9], and later Stienstra [35, 36], related the K-theory of endomorphisms to crystalline cohomology. More recently, work of Betley-Schlichtkrull [6], Hesselholt [25], and Lindenstrauss-McCarthy [29], establishes precise connections between the K-theory of (parametrized) endomorphisms, Goodwillie calculus, and trace methods in algebraic K-theory. In this article, we study the more foundational aspects of the K-theory of endomorphisms using noncommutative motives.

Noncommutative motives. Let $\mathrm{Cat}_\infty^{\mathrm{perf}}$ be the ∞-category of small idempotent-complete stable ∞-categories; see §2.3. Standard examples are the ∞-category Perf_R of perfect modules over a ring spectrum R and the ∞-category Perf_X of perfect complexes for a scheme X. Recall from [7, 6.1] that a functor $E\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}$, with values in a stable presentable ∞-category $\mathcal{D}$, is called an *additive invariant* if it preserves filtered colimits and sends split-exact sequences of ∞-categories to (necessarily split) cofiber sequences of spectra. Examples include algebraic K-theory (see §2.6), topological Hochschild homology (THH), and topological cyclic homology (TC). In [7, §6] we have constructed the *universal additive invariant*

$$(1.1) \qquad\qquad \mathcal{U}_{\mathrm{add}}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{M}_{\mathrm{add}}\,.$$

Given any stable presentable ∞-category $\mathcal{D}$, there is an induced equivalence

$$(1.2) \qquad\qquad (\mathcal{U}_{\mathrm{add}})^*\colon \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathcal{D})$$

where the left-hand side denotes the ∞-category of colimit-preserving functors and the right-hand side the ∞-category of additive invariants. Because of this property, which is reminiscent of motives, $\mathcal{M}_{\mathrm{add}}$ is called the category of *noncommutative motives*. As with any stable ∞-category, $\mathcal{M}_{\mathrm{add}}$ carries a natural enrichment $\mathrm{Map}(-,-)$ in spectra; see [7, §4.2]. In [7, §7.3] we proved that for every $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ there is a natural equivalence of spectra

$$(1.3) \qquad\qquad \mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_\mathbb{S}), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq K(\mathcal{C})\,,$$

where $\mathbb{S}$ stands for the sphere spectrum. Intuitively speaking, algebraic K-theory becomes co-represented by the noncommutative motive associated to $\mathbb{S}$.

Statements of results. Given an ∞-category $\mathcal{C}$, we start by defining the ∞-*category* $\mathrm{End}(\mathcal{C})$ *of endomorphisms in* $\mathcal{C}$ as the functor ∞-category $\mathrm{End}(\mathcal{C}) := \mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C})$; see definition 3.5. By first restricting this construction to $\mathrm{Cat}_\infty^{\mathrm{perf}}$ and then by applying the algebraic K-theory functor, we obtain a well-defined K-theory of endomorphisms functor

$$(1.4) \qquad\qquad \mathrm{KEnd}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{S}_\infty$$

with values in the ∞-category of symmetric spectra. Our first main result characterizes this functor as follows:

Theorem 1.5. *(see theorem 3.10) The above functor* (1.4) *is an additive invariant. Moreover, for every* $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ *there is a natural equivalence of spectra*

$$(1.6) \qquad\qquad \mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq \mathrm{KEnd}(\mathcal{C})\,,$$

where $\mathbb{S}[t]$ *stands for the tensor algebra of* $\mathbb{S}$.

Intuitively speaking, theorem 1.5 shows us that the functor (1.4) not only descends to $\mathcal{M}_{\mathrm{add}}$ but moreover becomes co-represented by the noncommutative motive associated to $\mathbb{S}[t]$. By adding a "formal variable" t to $\mathbb{S}$ one passes then from

algebraic K-theory (1.3) to K-theory of endomorphisms (1.6). The following result justifies the correctness of our construction.

Theorem 1.7. *(see theorem 3.13) Given an ordinary ring A, the associated spectrum* $\mathrm{KEnd}(\mathrm{Perf}_{HA})$ *(where HA denotes the Eilenberg-MacLane ring spectrum of A) is naturally equivalent to* $\mathrm{KEnd}(\mathbf{P}(A))$.

As explained by Almkvist in [1, page 339], a very interesting problem in the K-theory of endomorphisms is the classification of all the natural transformations of the functor $A \mapsto \mathrm{KEnd}(\mathbf{P}(A))$. Classical examples include the Frobenius F_n and the Verschiebung V_n operations. This problem was studied in the particular case of the K_0-theory of endomorphisms by Hazewinkel [22] and many operations in the higher K-theory of endomorphisms were computed by Stienstra [35, 36]. In §5 we extend F_n and V_n to the ∞-categorical setting and (making use of theorem 1.5) solve the problem raised by Almkvist as follows: given a (commutative) ring A, let us write $W_0(A)$ for the multiplicative (abelian) group

$$(1.8) \qquad \left\{ \frac{1 + a_1 r + \cdots + a_i r^i + \cdots + a_n r^n}{1 + b_1 r + \cdots + b_j r^j + \cdots + b_m r^m} \,|\, a_i, b_j \in A \; n, m \geq 0 \right\}.$$

Theorem 1.9. *(see theorem 5.7) There is a canonical weak equivalence of spectra*

$$(1.10) \qquad \mathrm{Nat}(\mathrm{KEnd}, \mathrm{KEnd}) \simeq \mathrm{KEnd}(\mathrm{Perf}_{\mathbb{S}[t]}),$$

where Nat *stands for the spectrum of natural transformations. Moreover, the group* $\pi_0 \mathrm{Nat}(\mathrm{KEnd}, \mathrm{KEnd})$ *of natural transformations up to homotopy is isomorphic to*

$$(1.11) \qquad \pi_0 \mathrm{KEnd}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq \mathbb{Z} \oplus W_0(\mathbb{Z}[t]).$$

Furthermore, under the above identifications, the Frobenius operations F_n correspond to the elements $(1, 1 + r^n t)$ and the Verschiebung operations V_n to the elements $(n, 1 + rt^n)$.

Roughly speaking, Theorem 1.9 shows us that all the information concerning a natural transformation of the above functor (1.4) can be completely encoded in an integer plus a fraction between polynomials with constant term 1. It shows also us that the Frobenius (resp. the Verschiebung) operation is the "simplest one" with respect to the variable t (resp. r). In order to prove the above isomorphism (1.11) we have made the following computation which is of general interest:

Theorem 1.12. *(see theorem 4.15) For every* connective *ring spectrum R, one has an isomorphism $\pi_0 \mathrm{KEnd}(\mathrm{Perf}_R) \simeq K_0(\mathrm{End}(\mathbf{P}(\pi_0 R)))$ of abelian groups.*

One interesting feature of the K-theory of endomorphisms is its connection with Witt vectors. Given a *commutative* ring A, the *Witt ring* $W(A)$ of A is the abelian group of all power series of the form $1 + a_1 r + a_2 r^2 + \cdots$, with $a_i \in A$, endowed with the multiplication $*$ determined by the equality $(1 - a_1 r) * (1 - a_2 r) = (1 - a_1 a_2 r)$. The *rational* Witt ring $W_0(A) \subset W(A)$ of A consists of the elements of the form (1.8). As observed by Grayson [19], $W_0(A)$ is a dense λ-subring of $W(A)$ and hence $W(A)$ can be recovered from $W_0(A)$ by a completion procedure.

As proved in [8, §4], the standard symmetric monoidal structure $- \otimes^\vee -$ on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ can be extended to $\mathcal{M}_{\mathrm{add}}$ in a universal way making (1.1) symmetric monoidal. This additional structure allows us to recover the rational Witt ring (and hence the Witt ring) from the category of noncommutative motives as follows:

Theorem 1.13. *(see §6) The ring maps* $\mathbb{S}[t] \overset{t \mapsto t \wedge t}{\rightarrow} \mathbb{S}[t] \wedge \mathbb{S}[t]$ *and* $\mathbb{S}[t] \overset{t=1}{\rightarrow} \mathbb{S}$ *induce a counital coassociative cocommutative coalgebra structure on* $\mathrm{Perf}_{\mathbb{S}[t]} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$. *Moreover, the ring maps* $\mathbb{S} \to \mathbb{S}[t]$ *and* $\mathbb{S}[t] \overset{t=0}{\rightarrow} \mathbb{S}$ *give rise to a wedge sum decomposition* $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}) \vee \mathbb{W}_0$ *of counital coassociative cocommutative coalgebras in* $\mathcal{M}_{\mathrm{add}}$.

Using theorem 1.13, we then obtain a well-defined lax symmetric monoidal functor $\mathrm{Map}(\mathbb{W}_0, -)$ from $\mathcal{M}_{\mathrm{add}}$ to $\mathcal{S}_\infty$ which we call the *rational Witt ring spectrum functor*. This terminology is justified by the following agreement result:

Theorem 1.14. *(see theorem 6.3) For every ordinary commutative ring A one has a ring isomorphism*

$$(1.15) \qquad \pi_0 \mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{HA})) \simeq W_0(A).$$

Isomorphism (1.15) provides a conceptual characterization of the rational Witt ring, as the left-hand side is defined solely in terms of universal properties. Roughly speaking, by only keeping track of the "formal variable" t of $\mathbb{S}[t]$ one passes from K-theory of endomorphisms (1.6) to rational Witt vectors (1.15).

Finally, making use of Lurie's resolution of Mandell's conjecture (see [28, 8.1.2.6]), we obtain the following result:

Corollary 1.16. *(see corollary 6.8) Let R be an E_n ring spectrum. Then the associated rational Witt ring spectrum* $\mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_R))$ *is an E_{n-1} ring spectrum.*

Intuitively speaking, corollary 1.16 shows us that the rational Witt ring spectrum functor $\mathrm{Map}(\mathbb{W}_0, -)$ decreases commutativity by one.

2. Preliminaries

2.1. **Notations.** Given an ordinary ring A, we will denote by $\mathrm{Ch}(A)$ the category of cochain complexes of (right) A-modules. We will assume that $\mathrm{Ch}(A)$ is endowed with the projective Quillen model structure; see [26, §2.3]. The associated homotopy category, i.e., the derived category of A, will be denoted by $\mathcal{D}(A)$. We will write $\mathrm{perf}(A)$ for the category of *perfect* complexes of A-modules, i.e., the full subcategory of $\mathrm{Ch}(A)$ consisting of those complexes that become compact in the derived category $\mathcal{D}(A)$. We will write $\mathrm{Ch}^b(-)$ for the full subcategory of bounded complexes. Finally, we write Perf_A to denote the ∞-category of perfect complexes over A (and more generally the category of compact modules over a stable ∞-category).

2.2. **The homotopy theory of a Waldhausen category.** An important source of categories with weak equivalences is provided by Waldhausen categories. Recall from [39] that this consists of a category $\mathcal{C}$ endowed with a subcategory of weak equivalences $w\mathcal{C}$ and with a subcategory of cofibrations $\mathrm{cof}(\mathcal{C})$ such that the pushouts along cofibrations exist, the cobase change of a cofibration is a cofibration, and pushout along cofibrations are homotopy pushouts. Following [10], we will impose further hypotheses that afford control on the underlying homotopy category of a Waldhausen category. These come in two forms:

(1) We want to ensure that $\mathcal{C}$ has a homotopy calculus of left factions (HCLF) in the sense of Dwyer-Kan [12]. Categories with weak equivalences that have homotopy calculi of fractions admit concise and tractable models for the mapping spaces in the Dwyer-Kan simplicial localization $L\mathcal{C}$ [12, 13]. Specifically, we can use concise models of $L^H\mathcal{C}$ to represent the homotopy types of the mapping spaces as the nerves of certain categories of zig-zags [13].

For this purpose, we impose factorization hypotheses. Recall from [10, 2.6] the notion of a *Waldhausen category with functorial mapping cylinders*. Such cylinders allow a functorial factorization $A \to Tf \to B$ of every map $f\colon A \to B$. The map $A \to Tf$ is a cofibration and $Tf \to B$ comes equipped with a natural section $B \to Tf$ (which is a weak equivalence). We will say that a Waldhausen category admits functorial factorization if every map $f\colon A \to B$ admits a functorial factorization as a cofibration followed by a weak equivalence; note that this implies functorial mapping cylinders (by factoring the fold map $A \coprod A \xrightarrow{\nabla} A$). It is possible to weaken our factorization hypotheses to remove the hypotheses of functoriality; see for instance [10, §A]. In fact, it is enough to require functorial mapping cylinders for *weak cofibrations*, i.e., maps that are equivalent via a zig-zag of weak equivalences to a cofibration; see [10, §2.1]. Given functorial mapping cylinders for weak cofibrations, the category $\mathcal{C}$ has a homotopy calculus of left fractions; see [10, 5.5].

(2) We want to ensure the weak equivalences are compatible with the homotopy category of $\mathcal{C}$. Recall from [15] that a category $\mathcal{C}$ with weak equivalences is called *DHKS-saturated* if a map is a weak equivalence if and only if it is an isomorphism in the homotopy category [15]. In a Waldhausen category with functorial mapping cylinders for weak cofibrations, the property of being DHKS-saturated is equivalent to the weak equivalences satisfying the two out of six property [10, 6.4]. In fact, it suffices that the weak equivalences are closed under retracts (which is easier to check).

2.3. **∞-categories.** Throughout the article we will assume that the reader is familiar with the basics of the theory of ∞-categories. For technical convenience, we work in the setting of Joyal's quasicategories, but nothing about our work depends on any particular model of ∞-categories. Standard references for quasicategories material are [27, 28]. We briefly review a few aspects of the theory of particular note for our treatment.

There are a number of options for producing the "underlying" ∞-category of a category equipped with a notion of "weak equivalence". The most structured setting is that of a simplicial model category $\mathcal{C}$, where the ∞-category can be obtained by restricting to the full simplicial subcategory $\mathcal{C}^{\mathrm{cf}}$ of cofibrant-fibrant objects and then applying the simplicial nerve functor N. More generally, if $\mathcal{C}$ is a category equipped with a subcategory of weak equivalences $w\mathcal{C}$, the Dwyer-Kan simplicial localization $L\mathcal{C}$ provides a corresponding simplicial category, and then $\mathrm{N}((L\mathcal{C})^{\mathrm{fib}})$, where $(-)^{\mathrm{fib}}$ denotes fibrant replacement in simplicial categories, yields an associated ∞-category. Lurie has given a version of this approach in [28, §1.3.3]: we associate to a (not necessarily simplicial) category $\mathcal{C}$ with weak equivalences W an ∞-category $\mathrm{N}(\mathcal{C})[W^{-1}]$; when $\mathcal{C}$ is a model category, for functoriality reasons it is usually convenient to restrict to the cofibrant objects $\mathcal{C}^{\mathrm{c}}$ and consider $\mathrm{N}(\mathcal{C}^{\mathrm{c}})[W^{-1}]$.

Recall from [7, §2.2] the notions of *stable* ∞-category and *idempotent-complete* stable ∞-category, and from [7, §2.3] the notion of *compact* ∞-category. Let us denote by Cat_∞ the ∞-category of small ∞-categories, by $\mathrm{Cat}_\infty^{\mathrm{ex}}$ the ∞-category of small stable ∞-categories, and by $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the ∞-category of small idempotent-complete stable ∞-categories. As explained in [7, §2.2-2.3], the inclusions of subcategories $\mathrm{Cat}_\infty^{\mathrm{perf}} \subset \mathrm{Cat}_\infty^{\mathrm{ex}} \subset \mathrm{Cat}_\infty$ admit left adjoints

$$\mathrm{Stab} \colon \mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}_\infty^{\mathrm{ex}}, \qquad \mathrm{Idem} \colon \mathrm{Cat}_\infty^{\mathrm{ex}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{perf}}.$$

Note that the inclusion $\mathrm{Cat}_\infty^{\mathrm{ex}} \subset \mathrm{Cat}_\infty^{\mathrm{perf}}$ is fully faithful, but $\mathrm{Cat}_\infty^{\mathrm{ex}} \subset \mathrm{Cat}_\infty$ is not.

2.4. **Spectral categories.** Recall from [11, §2][34, Appendix A] that a small *spectral category* is a category enriched over the symmetric monoidal category $\mathcal{S}$ of symmetric spectra. As explained in [7, §3] there is a close connection between spectral categories and ∞-categories. The category of small spectral categories $\mathrm{Cat}_\mathcal{S}$ admits a Quillen model structure with weak equivalencse the Morita equivalences [37], and the ∞-category $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is equivalent to the ∞-category associated to this model category; see [7, 3.20-3.20]. Hence, in order to simplify the exposition, we will make no notational distinction between a spectral category (for instance a ring spectrum) and the associated ∞-category.

2.5. **Algebraic K-theory of exact and Waldhausen categories.** We assume the reader has some familiarity with Quillen's K-theory of exact categories [32, §2] and with Waldhausen's K-theory of categories with cofibrations and weak equivalences (Waldhausen categories) [39, §1]. For each of these classes of input data, we can associate a connective algebraic K-theory spectrum. Given an exact category, we can regard it as a Waldhausen category with weak equivalences the isomorphisms and cofibrations the admissible monomorphisms; see [39, §1.9] for the agreement between the two possible constructions of K-theory in this situation.

2.6. **Algebraic K-theory of ∞-categories.** In this section we quickly review the analogue of the definition of Waldhausen K-theory in the setting of ∞-categories.

Definition 2.1. Let $\mathcal{C}$ be a pointed ∞-category. We say that $\mathcal{C}$ is an ∞-category with cofibrations if we have the additional data of a subcategory $\mathrm{cof}(\mathcal{C})$ such that

    (1) for any object x in $\mathcal{C}$, the unique map $* \to x$ is a cofibration,
    (2) the subcategory $\mathrm{cof}(\mathcal{C})$ contains all the equivalences,
    (3) pushouts along cofibrations exist, and the cobase change of a cofibration is again a cofibration.

We refer to the maps in $\mathrm{cof}(\mathcal{C})$ as cofibrations.

Such an ∞-category with cofibrations is referred to as a Waldhausen ∞-category in [4, 17]. Specifying a subcategory of cofibrations is a way of specifying which maps have homotopy cofibers. In many natural examples, all maps are cofibrations. For instance, any pointed category with pushouts admits the structure of an ∞-category with cofibrations where all maps are cofibrations.

There is a close connection between cofibrations in Waldhausen ∞-categories and the weak cofibrations studied in [10] (and reviewed in § 2.2 above). More precisely, we have the following consistency check:

Lemma 2.2. *Let $\mathcal{C}$ be a Waldhausen category with weak equivalences $w\mathcal{C}$ and cofibrations $\mathrm{cof}(\mathcal{C})$ that satisfy the hypothesis of factorization of weak cofibrations. Let*

us denote by $\mathrm{cof}(\mathrm{N}\mathcal{C}[W^{-1}]) \subseteq \mathrm{N}\mathcal{C}[W^{-1}]$ *the subcategory of those arrows which are equivalent to the image of an arrow in* $\mathrm{N}\,\mathrm{cof}(\mathcal{C}) \subseteq \mathrm{N}\mathcal{C}$. *Under these assumptions and notations,* $(\mathrm{N}\mathcal{C}[W^{-1}], \mathrm{cof}(\mathrm{N}\mathcal{C}[W^{-1}]))$ *is an* ∞-*category with cofibrations.*

Proof. By [10, 6.2], the factorization hypothesis implies that $\mathcal{C}$ has a homotopy calculus of left fractions. Then, using the universal property of the pushout in an ∞-category, we can conclude that pushouts along cofibrations compute pushouts in the underlying ∞-category. □

Remark 2.3. In particular, a Waldhausen category with factorization has an underlying ∞-category with cofibrations where all maps are cofibrations (and all pushouts exist).

Generalizing [28, 1.2.2.2], we have the following version of the $S_\bullet$ construction:

Definition 2.4. Let $(\mathcal{C}, \mathrm{cof}(\mathcal{C}))$ be an ∞-category with cofibrations. Denote by $\mathrm{Gap}([n], \mathcal{C}, \mathrm{cof}(\mathcal{C}))$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathrm{Ar}[n]), \mathcal{C})$ spanned by the functors $\mathrm{N}(\mathrm{Ar}[n]) \to \mathcal{C}$ such that, for each $i \in I$, $F(i, i)$ is a zero object of $\mathcal{C}$, $F(i, j) \to F(i, k)$ is a cofibration for $i \leq j \leq k$, and for each $i < j < k$, the following square is cocartesian

$$\begin{array}{ccc} F(i,j) & \longrightarrow & F(i,k) \\ \downarrow & & \downarrow \\ F(j,j) & \longrightarrow & F(j,k) \,. \end{array}$$

Following [28, 1.2.2.5], we define a simplicial ∞-category $S_\bullet^\infty \mathcal{C}$ by the rule $S_n^\infty \mathcal{C} = \mathrm{Gap}([n], \mathcal{C}, \mathrm{cof}(\mathcal{C}))$. Applying passage to the largest Kan complex levelwise, we obtain a simplicial space $(S_\bullet^\infty \mathcal{C})_{\mathrm{iso}}$. Then $\Omega|(S_\bullet^\infty \mathcal{C})_{\mathrm{iso}}|$ is the ∞-categorical version of Waldhausen's K-theory space. Furthermore, for each n, $\mathrm{Gap}([n], \mathcal{C}, \mathrm{cof}(\mathcal{C}))$ is itself equipped with a suitable subcategory of cofibrations: we can can iterate this procedure. Since $\mathrm{Gap}([0], \mathcal{C}, \mathrm{cof}(\mathcal{C}))$ is contractible (with preferred basepoint given by the point in $\mathcal{C}$) and $\mathrm{Gap}([1], \mathcal{C}, \mathrm{cof}(\mathcal{C}))$ is equivalent to $\mathcal{C}$, there is a natural map

$$S^1 \wedge (\mathcal{C})_{\mathrm{iso}} \longrightarrow |(S_\bullet^\infty \mathcal{C})_{\mathrm{iso}}|$$

given by the inclusion into the 1-skeleton. Therefore, the spaces $|((S_\bullet^\infty)^n(\mathcal{C}))_{\mathrm{iso}}|$ assemble to form a spectrum $K(\mathcal{C})$, which is evidently the ∞-categorical analogue of Waldhausen's K-theory spectrum. Furthermore, it is natural in functors of ∞-categories with cofibrations, i.e., functors which preserve zero objects, cofibrations, and pushouts along cofibrations.

The following results connect this definition to the usual definition. This is a generalization of the comparison of [7, 7.12], which handles the case when all maps are cofibrations.

Theorem 2.5. *Let* $\mathcal{C}$ *be a Waldhausen category with weak equivalences* $w\mathcal{C}$ *and cofibrations* $\mathrm{cof}(\mathcal{C})$. *Then there is a natural zig-zag of equivalences connecting* $K(\mathcal{C})$ *and* $K(\mathrm{N}\mathcal{C}[W^{-1}])$.

Proof. This follows from the rectification technique of [7, 7.11] coupled with the work of [7, 7.7]. □

2.7. **Grothendieck group of ∞-categories.** In this last subsection we give an explicit description of the Grothendieck group (K_0) of an ∞-category $\mathcal{C}$ with cofibrations. Observe that we can describe $S_2^\infty\mathcal{C}$ as the full subcategory of the ∞-category of 2-simplices $\sigma\colon \Delta^2 \to \mathcal{C}$ with the property that the composite

$$\Delta^{\{0,2\}} \longrightarrow \Delta^2 \longrightarrow \mathcal{C}$$

is equivalent to zero and the map specified by $\Delta^{\{0,1\}}$ is a cofibration. Moreover, definition 2.1 implies that the subcategory $S_2^\infty(\mathcal{C})$ always contains the "split exact sequence" $A \to A \oplus B \to B$ corresponding to each pair of objects A and B of $\mathcal{C}$.

Lemma 2.6. *The abelian group $K_0(\mathcal{C})$ can be described as the cokernel*

$$(2.7) \qquad \bigoplus_{\pi_0 S_2^\infty(\mathcal{C})^\simeq} \mathbb{Z} \longrightarrow \bigoplus_{\pi_0 \mathcal{C}^\simeq} \mathbb{Z} \longrightarrow K_0(\mathcal{C})$$

of the map which, on the component corresponding to the equivalence class of the exact sequence $[A \to B \to C] \in \pi_0 S_2(\mathcal{C})$, sends $1 \in \mathbb{Z}$ to the element $[A \oplus C] - [B] \in \bigoplus_{\pi_0 \mathcal{C}} \mathbb{Z}$. Here $\pi_0 \mathcal{C}^\simeq$ denotes the set of equivalence classes of objects of the ∞-category $\mathcal{C}$, which is to say the set of connected components of the underlying ∞-groupoid $\mathcal{C}^\simeq$ of $\mathcal{C}$.

Proof. The argument is similar to the classical analysis of K_0 in the Waldhausen context. $\qquad\square$

Example 2.8. If $\mathcal{C}$ is a stable ∞-category, viewed as an ∞-category with cofibrations by taking all arrows to be cofibrations, we see that $S_2^\infty(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ consisting of the (co)fiber sequences; that is, those $\sigma\colon \Delta^2 \to \mathcal{C}$ which extend to a cocartesian square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \searrow^{\sigma} & \downarrow \\ 0 & \longrightarrow & C \end{array}$$

where 0 denotes a zero object of $\mathcal{C}$. Equivalently, these are the 2-simplices which give rise to distinguished triangles $A \to B \to C$ in the underlying triangulated homotopy category.

3. K-theory of endomorphisms

In this section we extend the K-theory of endomorphisms from ordinary rings to stable ∞-categories. The main results are theorems 3.10-3.13 and proposition 3.11.

Definition 3.1. Let us denote by $\mathcal{D} \simeq \underline{\mathbb{N}}$ the category with one object and endomorphism monoid $\mathbb{N}$. The category of endomorphisms in a category $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$; that is, it has as objects the pairs (x, α) where $x \in \mathcal{C}$ and $\alpha\colon x \to x$ an endomorphism, and morphisms $(x, \alpha) \to (x', \alpha')$ the maps $f\colon x \to x'$ such that $f\alpha = \alpha' f$. Note that the unique functor $* \to \underline{\mathbb{N}}$ induces a forgetful functor $\mathrm{End}(\mathcal{C}) \to \mathcal{C}$.

3.1. **Exact categories.** Given an exact category $\mathcal{C}$ in the sense of Quillen [32, §2], the category $\mathrm{End}(\mathcal{C})$ inherits an exact structure by declaring a sequence to be exact if its image under $\mathrm{End}(\mathcal{C}) \to \mathcal{C}$ is exact. The endomorphism K-theory of $\mathcal{C}$ is then defined as the connective algebraic K-theory spectrum $\mathrm{KEnd}(\mathcal{C})$ of the exact category $\mathrm{End}(\mathcal{C})$. Clearly this construction is functorial in exact functors.

Given an ordinary ring A, let $\mathbf{P}(A)$ denote the exact category of finitely-generated projective (right) A-modules. Following Almkvist [1, 2] and Grayson [19], the K-theory of endomorphisms of A is defined as the K-groups associated to the spectrum $\mathrm{KEnd}(\mathbf{P}(A))$.

3.2. **Waldhausen categories.** We now extend the above construction to the setting of Waldhausen categories.

Lemma 3.2. *The category $\mathrm{End}(\mathcal{C})$ of endomorphisms in a Waldhausen category $\mathcal{C}$ carries a canonical Waldhausen structure. A morphism is a cofibration (resp. a weak equivalence) in $\mathrm{End}(\mathcal{C})$ if its image under $\mathrm{End}(\mathcal{C}) \to \mathcal{C}$ is a cofibration (resp. a weak equivalence) in $\mathcal{C}$. Furthermore, if $\mathcal{C}$ admits functorial factorization, then so does $\mathrm{End}(\mathcal{C})$.*

Proof. Pushouts along cofibrations in $\mathrm{End}(\mathcal{C})$ are computed in $\mathcal{C}$, using the induced endomorphism on the pushout. Hence, pushouts along cofibrations exist in $\mathrm{End}(\mathcal{C})$ and the gluing axiom holds. The remaining properties are clear. $\square$

Using Lemma 3.2, one can associate (as in the case of exact categories) to every Waldhausen category $\mathcal{C}$ a well-defined algebraic K-theory spectrum $\mathrm{KEnd}(\mathcal{C})$. This construction is clearly functorial in exact functors of Waldhausen categories.

Example 3.3. A motivating example is the case in which $\mathcal{C} = \mathrm{perf}(A)$ is the category of perfect complexes over an ordinary ring A. The weak equivalences are the quasi-isomorphisms and the cofibrations are the morphisms which admit retractions as morphisms of graded (right) A-modules (i.e., the degree-wise split monomorphisms). We have an equivalence of ∞-categories

$$\mathrm{N}(\mathrm{perf}(A))[W^{-1}] \simeq \mathrm{Perf}_{HA},$$

where HA is the associated ring spectrum. The category $\mathrm{End}(\mathrm{perf}(A))$ admits an algebraic description. Specifically, $\mathrm{End}(\mathrm{perf}(A))$ is the category of those complexes of $A[t]$-modules that are perfect as underlying complexes of A-modules. Note that $\mathrm{End}(\mathrm{perf}(A)) \neq \mathrm{perf}(A[t])$ since perfect complexes of $A[t]$-modules tend not to be perfect as complexes of A-modules.

Example 3.4. Example 3.3 can be generalized to the case where $\mathcal{C}$ is the category Perf_R of perfect complexes over a ring spectrum R. When R is the Eilenberg-MacLane spectrum HA of an ordinary ring A, we recover example 3.3. Note that in this case $\mathrm{End}(\mathrm{Perf}_R)$ also admits an algebraic description. Specifically, an R-module M endowed with an endomorphism is precisely the same data as a module over $R[\mathbb{N}] := R \wedge \Sigma_+^\infty \mathbb{N}$. This follows from the fact that $R \wedge \Sigma_+^\infty \mathbb{N}$ is the free R-algebra on one generator, so that a map to $\mathrm{End}(M)$ is (by adjunction) just a map of R-modules $M \to M$ (see [16, II.4.4] or [33, 3.10]).

3.3. **Endomorphisms of ∞-categories and co-representability.** The category $\underline{\mathbb{N}}$ with one object and endomorphism monoid $\mathbb{N}$ (under addition) is freely generated by a single nonidentity arrow. As a consequence, we obtain a well-defined map $\Delta^1/\partial\Delta^1 \to \mathrm{N}(\underline{\mathbb{N}})$ which is a categorical equivalence of simplicial sets.

Definition 3.5. Let $\mathcal{C} \in \mathrm{Cat}_\infty$ be an ∞-category.

(1) The ∞-category $\mathrm{End}(\mathcal{C})$ of endomorphisms in $\mathcal{C}$ is the functor ∞-category $\mathrm{End}(\mathcal{C}) := \mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathcal{C})$. Note that, as colimits in functor ∞-categories are computed pointwise, $\mathrm{End}(\mathcal{C})$ has finite colimits if and only if $\mathcal{C}$ has finite colimits. Moreover, if $\mathcal{C}$ is stable then $\mathrm{End}(\mathcal{C})$ is also stable [28, 1.1.3.1].

(2) If $\mathcal{C}$ has finite colimits and a zero object, then the *K-theory of endomorphisms of $\mathcal{C}$* is defined as the (connective) spectrum $\mathrm{KEnd}(\mathcal{C})$; see §2.6.

Clearly, definition 3.5(ii) is functorial in $\mathcal{C}$. Hence, we obtain a well-defined K-theory of endomorphisms functor

$$(3.6) \qquad\qquad \mathrm{KEnd}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow \mathcal{S}_\infty$$

with values in the ∞-category of symmetric spectra. In this section we prove that (3.6) not only descends to the category $\mathcal{M}_{\mathrm{add}}$ but moreover that it becomes corepresentable; see Theorem 3.10 below.

Notation 3.7. We denote by $\mathbb{S}[t]$ the tensor algebra on the sphere spectrum $\mathbb{S}$; see [33, Example 3.10].

Lemma 3.8. *The ∞-category $\mathrm{Perf}_{\mathbb{S}[t]} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ is compact.*

Proof. Recall that there is an equivalence $\mathrm{Cat}_\infty^{\mathrm{perf}} \simeq \mathrm{Pr}_{\mathrm{St},\omega}^{\mathrm{L}}$ induced by passage to the Ind-category [27, §5.5.7], where $\mathrm{Pr}_{St,\omega}^L$ is the ∞-category of compactly-generated stable ∞-categories. Thus, it suffices to show that the ∞-category of $\mathbb{S}[t]$-modules is compact as a compactly-generated stable ∞-category, i.e. as an object of $\mathrm{Pr}_{\mathrm{St},\omega}^{\mathrm{L}}$. By proposition 3.5 of [3], this reduces to showing that $\mathbb{S}[t]$ is compact as an $\mathbb{S}$-algebra, which is clear because it is free on one generator. $\square$

Proposition 3.9. *Let $\mathcal{A}$ be an object in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Then there is a natural equivalence*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{A}) \simeq \mathrm{End}(\mathcal{A}).$$

Proof. By the work of [7, §4], we know that $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{A})$ can be described as the ∞-category associated to the spectral category $\mathrm{rep}(\mathbb{S}[t], \tilde{\mathcal{A}})$ of $S[t]$-$\tilde{\mathcal{A}}$-bimodules, where $\tilde{\mathcal{A}}$ is a spectral lift of $\mathcal{A}$. An $\mathbb{S}[t]$-$\tilde{\mathcal{A}}$-bimodule is the same thing as a $\tilde{\mathcal{A}}$-module with an endomorphism. Next, the condition of being right compact means that these are precisely the $\tilde{\mathcal{A}}$-modules with endomorphisms that are compact as $\tilde{\mathcal{A}}$-modules; i.e., this is the category $\mathrm{End}(\mathrm{perf}(\tilde{\mathcal{A}}))$. $\square$

Theorem 3.10. *The above functor (3.6) is an additive invariant. Moreover, for every $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ there is a natural weak equivalence of spectra*

$$\mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq \mathrm{KEnd}(\mathcal{C})\,.$$

Proof. The first claim follows from the second. We can verify the second claim as follows. Since by Lemma 3.8 the ∞-category $\mathrm{Perf}_{\mathbb{S}[t]}$ is compact, it follows from [7, 7.13] that we have a natural equivalence of spectra

$$\mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{U}_{\mathrm{add}}(\mathcal{A})) \simeq K\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{A})\,.$$

Now we apply Proposition 3.9. To complete the comparison, we need to consider the Waldhausen structures giving rise to the algebraic K-theory of $\mathrm{rep}(S[t], \tilde{\mathcal{A}})$ and $\mathrm{KEnd}(\mathrm{perf}(\tilde{\mathcal{A}}))$ — Proposition 3.11 allows us to use the Waldhausen models for the K-theory spectra.

The Waldhausen structure on $\text{End}(\tilde{\mathcal{A}})$ is inherited from the forgetful functor $\text{End}(\text{perf}(\tilde{\mathcal{A}})) \to \text{perf}(\tilde{\mathcal{A}})$; cofibrations are maps which are cofibrations in $\text{perf}(\tilde{\mathcal{A}})$. On the other hand, the Waldhausen structure on $\text{rep}(S[t], \mathcal{A})$ is given by maps which are cofibrations of bimodules. The identity functor $\text{rep}(S[t], \tilde{\mathcal{A}}) \to \text{End}(\text{perf}(\tilde{\mathcal{A}}))$ is exact, and evidently induces an equivalence on homotopy categories. Since in both Waldhausen structures all maps are weak cofibrations and we have functorial factorization, the generalized version of the approximation theorem (e.g., see [10, 1.1]) implies that this exact functor induces an equivalence on K-theory spectra. $\square$

3.4. **Agreement property.** We start by showing that definition 3.5 subsumes the setting of Waldhausen categories.

Proposition 3.11. *(Agreement I) Let $\mathcal{C}$ be a DHKS-saturated Waldhausen category that admits functorial factorization and $\text{N}\mathcal{C}[W^{-1}]$ the associated ∞-category. Then there is a natural zig-zag of weak equivalences of spectra between $\text{KEnd}(\mathcal{C})$ and $\text{KEnd}(\text{N}\mathcal{C}[W^{-1}])$.*

Proof. First, assume that $\mathcal{C}$ is a full subcategory of the cofibrant objects in a combinatorial pointed model category $\mathcal{A}$. Then [28, 1.3.4.25] implies that there is an equivalence

$$\text{N}(\text{Fun}(\underline{\mathbb{N}}, \mathcal{C})[W^{-1}] \simeq \text{Fun}(\Delta^1/\partial\Delta^1, \text{N}\mathcal{C}[W^{-1}])$$

and therefore an equivalence of spectra

$$(3.12) \qquad K(\text{N}(\text{Fun}(\underline{\mathbb{N}}, \mathcal{C})[W^{-1}]) \simeq K\,\text{End}(\text{N}\mathcal{C}[W^{-1}]).$$

Although the Waldhausen structure on $\text{End}(\mathcal{C})$ need not arise as induced from a model structure by restriction to the cofibrant objects, since $\text{End}(\mathcal{C})$ inherits functorial factorization from $\mathcal{C}$ we can reduce to this case [7, 7.7]. Next, using [7, 7.10], which compares the K-theory of a Waldhausen category to the K-theory of the underlying ∞-category, we find that the left-hand side of (3.12) is equivalent to the Waldhausen K-theory $\text{KEnd}(\mathcal{C})$. When $\mathcal{C}$ is an arbitrary Waldhausen category that admits functorial factorization and is DHKS-saturated, we can again use [7, 7.10] to reduce to the case of a full subcategory of the cofibrant objects in a pointed model category. $\square$

The following result relates the classical definition of the K-theory of endomorphisms of an ordinary ring A with the definition given herein for the associated Eilenberg-Mac Lane spectrum HA.

Theorem 3.13. *(Agreement II) Let A be an ordinary ring. There exists a canonical zig-zag of weak equivalences of spectra between the K-theory spectrum $\text{KEnd}(\mathbf{P}(A))$ of the exact category $\text{End}(\mathbf{P}(A))$ and the K-theory spectrum $\text{KEnd}(\text{Perf}_{HA})$ of the ∞-category $\text{End}(\text{Perf}_{HA})$.*

Proof. Using Proposition 3.11, we have a zig-zag of weak equivalences between $\text{KEnd}(\text{Perf}_{HA})$ and $\text{KEnd}(\text{perf}(HA))$. We can assume without loss of generality that HA is a cofibrant S-algebra. Recall from the discussion of Example 3.4 that $\text{End}(\text{perf}(HA))$ is equivalent to the category of $HA \wedge \mathbb{N}_+$-modules that are perfect as HA-modules. Moreover, $HA \wedge \mathbb{N}_+ \cong HA[t]$. By [16, §IV.2.4], the derived category $\text{Ho}(HA[t])$ of $HA[t]$-modules is equivalent to the derived category $\text{Ho}(A[t])$

of complexes of $A[t]$-modules. More generally, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ho}(HA[t]) & \xrightarrow{\simeq} & \mathrm{Ho}(A[t]) \\
\downarrow & & \downarrow \\
\mathrm{Ho}(HA) & \xrightarrow{\simeq} & \mathrm{Ho}(A),
\end{array}$$

where the vertical maps are induced from the forgetful functors. As a consequence, we can conclude that there is an equivalence of homotopy categories between $\mathrm{End}(\mathrm{perf}(HA))$ and the Waldhausen category $\mathrm{End}(\mathrm{perf}(A))$ where $\mathrm{perf}(A)$ denotes the Waldhausen category of perfect complexes of A-modules, where the cofibrations are the split monomorphisms and the weak equivalences the quasi-isomorphisms.

Now, using the identification (where Ch^b stands for bounded complexes)

$$\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A))) \cong \mathrm{Ch}^b(\mathrm{End}(\mathbf{P}(A))),$$

we obtain a zig-zag of exact functors connecting $\mathrm{End}(\mathbf{P}(A))$ to $\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A))$. Specifically, let $\mathcal{E}'$ denote the Waldhausen category structure on $\mathrm{Ch}^b(\mathrm{End}(\mathbf{P}(A)))$ where the cofibrations are the levelwise admissible monomorphisms and the weak equivalences are the quasi-isomorphisms. Then by [38, 1.11.7], the zig-zag of exact functors

$$\mathrm{End}(\mathbf{P}(A)) \longrightarrow \mathcal{E}' \xleftarrow{\mathrm{Id}} \mathrm{Ch}^b(\mathrm{End}(\mathbf{P}(A)))$$

induces equivalences on passage to K-theory. Therefore, it suffices to show that the exact inclusion functor

$$\iota\colon \mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A))) \longrightarrow \mathrm{End}(\mathrm{perf}(A))$$

induces an equivalence on K-theory spectra. We do this using a modern reformulation of [38, 1.11.7] arising from Barwick's theory of the K-theory of exact ∞-categories [5]. Since Lemma 3.14 below implies that $\mathrm{Ho}(\mathrm{End}(\mathrm{perf}(A)))$ has a bounded t-structure, we can apply [5, 5.6.1] to compare $K(\mathrm{End}(\mathrm{perf}(A)))$ to the Quillen K-theory of the heart of the t-structure regarded as an exact category. As the heart is precisely $\mathrm{End}(\mathbf{P}(A))$, the fact that this equivalence factors through ι shows that ι is an equivalence as well. $\qquad\square$

We now establish the existence of a bounded t-structure on $\mathrm{End}(\mathrm{perf}(A))$. Recall that a t-structure on a triangulated category $\mathcal{C}$ is determined by a pair of full subcategories $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$ such that:

(1) For objects X, Y in $\mathcal{C}$ such that $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y[-1]) = 0$.
(2) There are inclusions $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$.
(3) For any object X in $\mathcal{C}$, there exists a distinguished triangle $X' \to X \to X''$ such that $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq 0}[-1]$.

A standard example of a t-structure is given by considering $\mathrm{Ho}(\mathrm{perf}(A))$ for an ordinary ring A (the derived category of perfect complexes), and defining $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ to be the complexes with non-negative homology and non-positive homology, respectively.

We write $\mathcal{C}_{\leq n}$ for $\mathcal{C}_{\leq 0}[n]$ and $\mathcal{C}_{\geq m}$ for $\mathcal{C}_{\geq 0}[m]$. A t-structure is bounded if all objects in $\mathcal{C}$ are contained in $\mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq m}$ for some $n > m$. The standard t-structure on $\mathrm{Ho}(\mathrm{perf}(A))$ is evidently bounded.

Lemma 3.14. *Let A be an ordinary ring. Then the category*

$$\mathrm{Ho}(\mathrm{End}(\mathrm{perf}(A))) \simeq \mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_{HA}))$$

is triangulated and has a bounded t-structure induced from the triangulation and standard bounded t-structure on $\mathrm{Ho}(\mathrm{perf}(A))$.

Proof. Recall that the homotopy category of a stable ∞-category is triangulated [28, 1.1.2.14]. Since $\mathrm{End}(\mathrm{Perf}_{HA})$ inherits the structure of a stable ∞-category from Perf_{HA} [28, 1.1.3.1], $\mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_{HA}))$ has a triangulation in which the distinguished triangles in $\mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_{HA}))$ are induced by the forgetful functor

$$\mathrm{Ho}(\mathrm{End}(\mathrm{Perf}_{HA})) \longrightarrow \mathrm{Ho}(\mathrm{Perf}_{HA}).$$

Since the underlying stable ∞-category of $\mathrm{perf}(A)$ is equivalent to Perf_{HA}, this triangulation on Perf_{HA} can be regarded as coming from the triangulation on $\mathrm{perf}(A)$.

The t-structure on $\mathrm{Ho}(\mathrm{perf}(A))$ induces one on $\mathrm{Ho}(\mathrm{End}(\mathrm{perf}(A)))$ using full subcategories $\mathrm{Ho}(\mathrm{End}(\mathrm{perf}(A)))_{\geq 0}$ and $\mathrm{Ho}(\mathrm{End}(\mathrm{perf}(A)))_{\leq 0}$ determined by the forgetful functor. The only nontrivial condition to check is that any X can be fit into a triangle $X' \to X \to X''$ where X' has nonnegative homology and X'' has nonpositive homology. For any connective ring spectrum R, there is a functorial construction of the connective cover on the category of R-modules, such that we have a natural transformation $C \to \mathrm{id}$ [30, 4.2]. Since it is functorial, this construction passes to $\mathrm{End}(\mathrm{perf}(A))$, and the associated cofiber sequence gives the required triangle.

Finally, it is clear that the induced t-structure on $\mathrm{Ho}(\mathrm{End}(\mathrm{perf}(A)))$ is bounded since the one on $\mathrm{Ho}(\mathrm{perf}(A))$ is. $\qquad\square$

We conclude the section with a technical lemma which gives a partial analysis of the homotopy category of $\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A)))$.

Lemma 3.15. *The induced functor*

$$(3.16) \qquad\qquad \iota \colon \mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A))) \longrightarrow \mathrm{End}(\mathrm{perf}(A)),$$

is homotopically essentially surjective, where we equip each side with the weak equivalences given by the underlying quasi-isomorphisms.

Proof. We will decorate quasi-isomorphisms with the symbol $\sim$. Recall that we have fully-faithful inclusions $\mathrm{Ch}^b(\mathbf{P}(A)) \hookrightarrow \mathrm{perf}(A) \hookrightarrow \mathrm{Ch}(A)$ and that $\mathrm{Ch}(A)$ carries a projective Quillen model structure in which every object is fibrant [26, 2.3.11]. The associated homotopy category is the derived category $\mathcal{D}(A)$ of A.

Let $(M^\bullet, \alpha)$ be an object of $\mathrm{End}(\mathrm{perf}(A))$, i.e., a complex $M^\bullet \in \mathrm{perf}(A)$ and an endomorphism α of $M^\bullet$. Since $M^\bullet \in \mathrm{perf}(A)$, $M^\bullet$ has the homotopy type of a wedge summand of a finite cell complex of A-modules; that is, there exists an isomorphism in $\mathcal{D}(A)$ between $M^\bullet$ and a complex $P^\bullet \in \mathrm{Ch}^b(\mathbf{P}(A))$. All the objects of $\mathrm{Ch}^b(\mathbf{P}(A))$ are cofibrant [26, 2.3.6] and so this isomorphism lifts to a quasi-isomorphism $\theta \colon P^\bullet \xrightarrow{\sim} M^\bullet$. Associated to α we obtain then a well-defined endomorphism of $P^\bullet$ in the derived category $\mathcal{D}(A)$. Since $P^\bullet$ is cofibrant we can choose a representative $\overline{\alpha} \colon P^\bullet \to P^\bullet$ of this endomorphism. We obtain then a square

(3.17)
$$
\begin{array}{ccc}
P^\bullet & \xrightarrow[\sim]{\theta} & M^\bullet \\
\overline{\alpha}\downarrow & & \downarrow\alpha \\
P^\bullet & \xrightarrow[\theta]{\sim} & M^\bullet
\end{array}
$$

which is commutative only in the derived category $\mathcal{D}(A)$. The proof will consist now on replacing the quasi-isomorphism θ in (3.17) by a zig-zag of strictly commutative squares relating $\overline{\alpha}$ to α. Let

$$ P^\bullet \stackrel{\sim}{\rightarrowtail} X^\bullet \stackrel{\sim}{\twoheadrightarrow} M^\bullet $$

be a factorization of θ provided by the projective model structure. Note that $X^\bullet$ is cofibrant since this is the case of $P^\bullet$. Moreover, it belongs to $\mathrm{perf}(A)$ since it is quasi-isomorphic to $M^\bullet$. The lifting properties of the projective model structure furnish us morphisms $\overline{\beta}$ and β making the following two diagrams commutative

(3.18)
$$
\begin{array}{ccc}
P^\bullet & \stackrel{\sim}{\rightarrowtail} & X_\bullet \\
\overline{\alpha}\downarrow & & \downarrow\overline{\beta} \\
P^\bullet & \rightarrowtail[\sim] & X^\bullet
\end{array}
\qquad
\begin{array}{ccc}
X_\bullet & \stackrel{\sim}{\twoheadrightarrow} & M^\bullet \\
\beta\downarrow & & \downarrow\alpha \\
X^\bullet & \twoheadrightarrow[\sim] & M^\bullet.
\end{array}
$$

By combining the squares (3.17)-(3.18), we conclude then that the endomorphisms $\overline{\beta}$ and β of $X^\bullet$ agree in the derived category $\mathcal{D}(A)$. Since $X^\bullet$ is cofibrant object there exists a cylinder object

$$ X^\bullet \oplus X^\bullet \stackrel{[i_0\ i_1]}{\rightarrowtail} \mathrm{Cyl}(X^\bullet) \stackrel{\sim}{\twoheadrightarrow} X^\bullet $$

and a morphism H making the following diagram commute

$$
\begin{array}{ccc}
X^\bullet & \xrightarrow[\sim]{i_0} \mathrm{Cyl}(X^\bullet) \xleftarrow[\sim]{i_1} & X^\bullet \\
& \searrow^{\overline{\beta}} \quad \downarrow^{H} \quad \swarrow^{\beta} & \\
& X^\bullet &
\end{array}
\qquad .
$$

Note that since $\mathrm{Cyl}(X^\bullet)$ is quasi-isomorphic to X, it also belongs to $\mathrm{perf}(A)$. Consider the following commutative solid diagram:

(3.19)
$$
\begin{array}{ccc}
X^\bullet \oplus X^\bullet & \xrightarrow{\overline{\beta}\oplus\beta} X^\bullet \oplus X^\bullet \xrightarrow{[i_0\ i_1]} & \mathrm{Cyl}(X^\bullet) \\
{\scriptstyle [i_0\ i_1]}\downarrow & \quad\quad\quad \overset{\widetilde{H}}{\dashrightarrow} & \downarrow{\scriptstyle\sim} \\
\mathrm{Cyl}(X^\bullet) & \xrightarrow[H]{} & X^\bullet.
\end{array}
$$

By the lifting properties of the projective model structure there exists a well-defined morphism $\widetilde{H}$ as above making both triangles of the diagram commute. Now, consider the following commutative diagram:

(3.20)
$$
\begin{array}{ccccccccc}
P^\bullet & \overset{\sim}{\rightarrowtail} & X^\bullet & \overset{i_0}{\underset{\sim}{\longrightarrow}} & \mathrm{Cyl}(X) & \overset{i_1}{\underset{\sim}{\longleftarrow}} & X^\bullet & \overset{\sim}{\twoheadrightarrow} & M^\bullet \\
\downarrow{\overline{\alpha}} & & \downarrow{\overline{\beta}} & & \downarrow{\widetilde{H}} & & \downarrow{\beta} & & \downarrow{\alpha} \\
P^\bullet & \underset{\sim}{\rightarrowtail} & X^\bullet & \underset{i_0}{\overset{\sim}{\longrightarrow}} & \mathrm{Cyl}(X) & \underset{i_1}{\overset{\sim}{\longleftarrow}} & X^\bullet & \underset{\sim}{\twoheadrightarrow} & M^\bullet ,
\end{array}
$$

Note that the commutativity of the two interior squares is equivalent to the commutativity of the upper triangle in (3.19). The diagram (3.20) can then be interpreted as a zig-zag of quasi-isomorphisms in the category $\mathrm{End}(\mathrm{perf}(A))$ relating $(P^\bullet, \overline{\alpha})$ with $(M^\bullet, \alpha)$. As a consequence, $(P^\bullet, \overline{\alpha})$ and $(M^\bullet, \alpha)$ become isomorphic in the homotopy category $\mathrm{Ho}(\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A))))$. Since $(P^\bullet, \alpha)$ belongs to $\mathrm{End}(\mathrm{Ch}^b(\mathbf{P}(A)))$ the proof is then finished. $\qquad\square$

4. Endomorphisms of projective modules

In this section, we compute $\pi_0 \, \mathrm{KEnd}(\mathrm{Perf}_R)$ for a connective ring spectrum R; see theorem 4.15. To do this, we begin by rapidly reviewing the theory of free and projective modules over connective ring spectra; see [28, 8.2.2.4] for further details or [3, §2] for an exposition of the relevant results. Throughout this section, R will stand for a *connective* ring spectrum R.

Definition 4.1. An R-module M is *(finite) free* there exists a (finite) set I and an equivalence of R-modules $R^{\oplus I} \simeq M$. An R-module M is *(finite) shifted free* if there exists a (finite) $\mathbb{Z}$-graded set $I \to \mathbb{Z}$ and an R-module equivalence $\bigoplus_{n \in \mathbb{Z}} \Sigma^n R^{\oplus I_n} \simeq M$, where I_n denotes the fiber of $I \to \mathbb{Z}$ over $n \in \mathbb{Z}$.

Definition 4.2. An R-module P is said to be *projective* if it is projective as an object of the ∞-category $\mathrm{Mod}_R^{\geq 0}$ of connective R-modules, i.e., the functor

$$\mathrm{map}_R(P, -) \colon \mathrm{Mod}_R^{\geq 0} \longrightarrow \mathcal{T}$$

commutes with geometric realizations of connective R-modules.

Remark 4.3. We could give Definition 4.1 for modules over non-connective ring spectra, but the notion of shifted free module is really only sensible if R is not periodic; that is, if the only integer n for which there exists an equivalence $\Sigma^n R \to R$ is zero (of course, this property always holds for connective ring spectra). Similarly, Definition 4.2 implies that P is a connective R-module to start with; again, this notion of projective R-module is only properly behaved when R is connective. This is because, for an arbitrary ring spectrum R, connective or not, there are no nontrivial projective objects of the ∞-category Mod_R of all right R-modules (and of course the same is true in the ∞-category $\mathrm{Mod}_{R^{\mathrm{op}}}$ of left R-modules). See the argument immediately following [3, 2.5] for details.

Proposition 4.4. *Suppose that P is a connective R-module. Then the following are equivalent:*

    (1) *The R-module P is projective.*
    (2) *The R-module P is a retract of a free R-module.*

(3) *The functor*

$$\mathrm{map}_R(P, -)\colon \mathrm{Mod}_R^{\geq 0} \longrightarrow \mathcal{T}$$

preserves surjections (i.e., morphisms which are surjective on π_0).

(4) *Given a surjection (on π_0) of (not necessarily connective) R-modules $N \to M$ and any map $P \to M$, there exists a map $g\colon P \to N$ such that the resulting diagram*

$$
\begin{array}{ccc}
 & P & \\
 \swarrow & & \searrow \\
 N & \longrightarrow & M
\end{array}
$$

commutes in Mod_R.

Proof. See [28, §8]. Note that, in (4), we may assume without loss of generality that M and N are connective, as $\pi_0 \mathrm{map}(P, M) \cong \pi_0 \mathrm{map}(P, \tau_{\geq 0} M)$. $\square$

Proposition 4.4 shows that projective modules over connective ring spectra behave in much the same way as projective modules over ordinary rings. In particular, it motivates the next definition.

Definition 4.5. An R-module M is said to be *shifted projective* if it is a retract of a shifted free R-module.

We will only be interested in finite (shifted) projective R-modules.

Definition 4.6. A (shifted) projective R-module M is *finite (shifted) projective* if it is a retract of a finite (shifted) free R-module.

Note that finite (shifted) projective modules are perfect, since finite (shifted) free modules are perfect and retracts of compact objects are still compact. We write $\mathrm{Proj}_R \subset \mathrm{Perf}_R$ for the full subcategory of Perf_R consisting of the finite projective R-modules, and we write $\mathrm{Proj}_R^\Sigma \subset \mathrm{Perf}_R$ for the full subcategory of Perf_R consisting of the finite shifted projective R-modules. The following comparison provides a description of Proj_R in terms of the discrete ring $\pi_0 R$.

Proposition 4.7. *Let R be a connective ring spectrum. Then*

$$\pi_0\colon \mathrm{Ho}(\mathrm{Proj}_R) \longrightarrow \mathrm{Proj}_{\pi_0 R}$$

is an equivalence of categories.

Proof. This is standard. For a proof, see [3, 2.12], for instance. $\square$

The first step towards computing $\pi_0(\mathrm{KEnd}(\mathrm{Perf}_R))$ of this section is to prove that, for R a connective ring spectrum, there is an isomorphism $K_0(\mathrm{End}(\mathrm{Proj}_R)) \cong K_0(\mathrm{End}(\mathrm{Perf}_R))$. Here to define $K_0(\mathrm{End}(\mathrm{Proj}_R)) = \pi_0 \mathrm{KEnd}(\mathrm{Proj}_R))$ we will specify the structure of an ∞-category with cofibrations on $\mathrm{End}(\mathrm{Proj}_R)$. First, we specify the cofibrations on Proj_R as the maps $P \to P \coprod Q$ such that the cofiber (in Perf_R) is Q. It is straightforward to check that this definition satisfies the conditions of Definition 2.1. Then we define a map to be a cofibration in $\mathrm{End}(\mathrm{Proj}_R)$ if its image under the forgetful functor $\mathrm{End}(\mathrm{Proj}_R) \to \mathrm{Proj}_R$ is a cofibration.

We begin by recalling the analogous result in the setting without endomorphisms.

Theorem 4.8. *For any connective ring spectrum R, the map*

$$i\colon K_0(\mathrm{Proj}_R) \longrightarrow K_0(\mathrm{Perf}_R),$$

induced by the inclusion $\mathrm{Proj}_R \to \mathrm{Perf}_R$, is an isomorphism.

Proof. We define a map

$$j \colon K_0(\mathrm{Perf}_R) \longrightarrow K_0(\mathrm{Proj}_R)$$

and check that it is inverse to i. The idea is to show that any generator $[M]$ of $K_0(\mathrm{Perf}_R)$ is a signed sum of generators $[P]$ of $K_0(\mathrm{Proj}_R)$, and the proof is by induction on the length $l = b - a$ of the Tor-amplitude of M (see [21] or [38] for a discussion of Tor-amplitude in the setting of derived categories and [3, §2.4] for the analogous treatment in the setting of modules over a ring spectrum).

To this end, let M be a perfect R-module. By [3, 2.13.1], there exists integers $a \leq b$ such that M has Tor-amplitude contained in the interval $[a, b]$. If $b - a = 0$, then by [3, 2.13.6], $M \simeq \Sigma^a P$ for some finite projective R-module P, in which case

$$[M] = (-1)^a [P] \in K_0(\mathrm{Perf}_R)$$

and we define

$$j([M]) := (-1)^a [P] \in K_0(\mathrm{Proj}_R).$$

This preserves the K_0-relations amongst R-modules with Tor-amplitude of length 0, because any such comes from a (split) cofiber sequence of the form

$$\Sigma^a P \longrightarrow \Sigma^a (P \oplus Q) \longrightarrow \Sigma^a Q$$

for some finite projective R-modules P and Q and

$$[P] - [P \oplus Q] + [Q] = 0 \in K_0(\mathrm{Proj}_R).$$

Inductively, suppose that we have defined j on the subcategory of Perf_R consisting of those perfect R-modules with Tor-amplitude contained in an interval $[a, b]$ with $b - a < l$ for some positive integer l, and let M be a perfect R-module with Tor-amplitude contained in an interval $[a, b]$ with $b - a = l$. Then by [3, 2.13.7], there is a finite projective R-module P and a cofiber sequence

$$\Sigma^a P \longrightarrow M \longrightarrow N$$

such that the cofiber N has Tor-amplitude contained in $[a + 1, b]$. Since $\Sigma^a P$ and N are perfect with Tor-amplitude of length $b - a + 1 < l$, we set

$$(-1)^a j([P]) - j([M]) + j([N]) = 0 \in K_0(\mathrm{Proj}_R).$$

In other words, killing bottom homotopy groups by mapping in shifted projectives results in a length l filtration

$$M \simeq M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_l \longrightarrow M_{l+1} \simeq 0$$

in which the filtration quotients $M_{k+1}/M_k \simeq \Sigma^{a+k+1} P_i$, $0 \leq k \leq l$, are shifts of (possibly trivial) finite projective R-modules P_k.

A separate induction implies that the resulting map

$$j \colon \bigoplus_{M \in \pi_0 \mathrm{Perf}_R} \mathbb{Z} \longrightarrow K_0(\mathrm{Proj}_R)$$

respects the relations in K_0, as any relation amongst perfect R-modules of Tor-amplitude of length not more than l comes from a cofiber sequence

$$L \longrightarrow M \longrightarrow N,$$

and we may proceed on the length $k \leq l$ of the Tor-amplitude of L. Indeed, if $k = 0$ then L is a finite shifted projective, in which case the well-definedness is clear. If $k > 0$, then there is a finite projective P and a cofiber sequence $\Sigma^a P \to L \to L'$

such that L' has Tor-amplitude of length strictly less than k. Defining M' as the pushout of $L' \leftarrow L \rightarrow M$, we obtain a map of cofiber sequences

$$
\begin{array}{ccccc}
L & \longrightarrow & M & \longrightarrow & N \\
\downarrow & & \downarrow & & \downarrow \\
L' & \longrightarrow & M' & \longrightarrow & N
\end{array}
$$

in which the right vertical map is an equivalence. This gives the relation $[L]-[M] = [L']-[M']$ in $K_0(\mathrm{Perf}_R)$, so that

$$j([L]) - j([M]) - j([N]) = j([L']) - j([M']) + j([N]) = 0$$

by inductive hypothesis. Since any relation in $K_0(\mathrm{Perf}_R)$ necessarily only involves finitely many perfect R-modules and therefore has Tor-amplitude contained in some finite interval $[a,b]$, it follows that j is a well-defined homomorphism of abelian groups.

It remains to show that i and j are inverse isomorphisms. It is clear that for any finite projective R-module P, $j(i([P])) = [P]$; likewise, if M is a perfect R-module with Tor-amplitude contained in an interval $[a,b]$ of length $b - a = l$, then

$$i(j([M])) = (-1)^{a_0}i([P_0]) + \cdots + (-1)^{a_l}i([P_l]) = [\Sigma^{a_0}P_0] + \cdots + [\Sigma^{a_l}P_l] = [M]$$

since, by construction, M admits a finite filtration with quotients $\Sigma^{a_k}P_k$. $\qquad\square$

The argument above demonstrates that any perfect R-module admits a finite descending filtration whose quotients are finite shifted projectives. A similar argument, which is the content of the following theorem, shows that objects of $\mathrm{End}(\mathrm{Perf}_R)$ admit finite descending filtrations whose quotients are shifts of objects of $\mathrm{End}\,\mathrm{Proj}_R$.

Theorem 4.9. *For any connective ring spectrum R, the map*

$$i \colon K_0(\mathrm{End}(\mathrm{Proj}_R)) \longrightarrow \pi_0 K_0(\mathrm{End}(\mathrm{Perf}_R)),$$

induced by the inclusion $\mathrm{Proj}_R \to \mathrm{Perf}_R$, is an isomorphism.

Proof. As before, we define an analogous map

$$j \colon K_0(\mathrm{End}(\mathrm{Perf}_R)) \longrightarrow K_0(\mathrm{End}(\mathrm{Proj}_R))$$

and check that it is inverse to i. Again, the proof goes by induction on the length of the Tor-amplitude of a given generator $[\alpha \colon M \to M]$ of $K_0(\mathrm{End}(\mathrm{Perf}_R))$. If M has Tor-amplitude contained in $[a,b]$ with $a - b = 0$, then $M \simeq \Sigma^a P$ for some finite projective R-module P, and we define

$$j([M \xrightarrow{\alpha} M]) = (-1)^a[P \xrightarrow{e} P]$$

where $e \simeq \Sigma^{-a}\alpha \colon P \to P$ is the shifted endomorphism. This is well-defined because, given a map of cofiber sequences

$$
\begin{array}{ccccc}
P & \longrightarrow & P \oplus Q & \longrightarrow & Q \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \begin{pmatrix} e & g \\ h & f \end{pmatrix}} & & \downarrow{\scriptstyle f} \\
P & \longrightarrow & P \oplus Q & \longrightarrow & Q,
\end{array}
$$

we have that

$$[P \xrightarrow{e} P] - [P \oplus Q \xrightarrow{\begin{pmatrix} e & g \\ h & f \end{pmatrix}} P \oplus Q] + [Q \xrightarrow{f} Q] = 0 \in K_0(\mathrm{End}(\mathrm{Proj}_R)).$$

Inductively, suppose that j is defined for all $[M \xrightarrow{\alpha} M]$ such that M has Tor-amplitude contained in an interval $[a, b]$ of length not more than l. Let P be a finite projective and $\Sigma^a P \to M \to N$ a cofiber sequence such that N has Tor-amplitude contained in $[a+1, b]$ ([3, 2.7]). Taking homotopy, the resulting exact sequence

$$\pi_0(P) \longrightarrow \pi_0(\Sigma^{-a}M) \longrightarrow \pi_0(\Sigma^{-a}N)$$

shows that $P \to \Sigma^{-a}M$ is surjective on π_0, since $\pi_0(\Sigma^{-a}N) \cong 0$ by lemma 4.10 below. It follows from proposition 4.4 that there exists an endomorphism e of P making the lower triangle in the diagram

$$
\begin{array}{ccc}
P & \longrightarrow & \Sigma^{-a}M \\
\downarrow & \searrow & \downarrow \\
P & \longrightarrow & \Sigma^{-a}M
\end{array}
$$

commutes; that is, there is a map from $\Sigma^a P \xrightarrow{\Sigma^a e} \Sigma^a P$ to $M \xrightarrow{\alpha} M$ in $\mathrm{End}(\mathrm{Perf}_R)$. Writing $\beta \colon N \to N$ for the cofiber of this map in $\mathrm{End}(\mathrm{Perf}_R)$, we see that N is a perfect R-module with Tor-amplitude in the interval $[a+1, b]$. Repeating this process gives a filtration

$$
\begin{array}{ccccccccc}
M = M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_l & \longrightarrow & M_{l+1} \simeq 0 \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & & & \downarrow{\scriptstyle \alpha_l} & & \downarrow{\scriptstyle \alpha_{l+1}} \\
M = M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_l & \longrightarrow & M_{l+1} \simeq 0
\end{array}
$$

of $M \xrightarrow{\alpha} M$ in $\mathrm{End}\,\mathrm{Perf}_R$ in which $\alpha_0 \simeq \alpha$ and $\alpha_{l+1} \simeq 0$, and the filtration quotients are of the form

$$\Sigma^{a+k+1} P_k \xrightarrow{\Sigma^{a+k} e_k} \Sigma^{a+k+1} P_k$$

for some $P_k \xrightarrow{e_k} P_k$ in $\mathrm{End}\,\mathrm{Proj}_R$, $0 \leq k \leq l$. The same argument as before shows that this is well-defined, and that the resulting homomorpism $j \colon K_0(\mathrm{End}\,\mathrm{Perf}_R) \to K_0(\mathrm{End}\,\mathrm{Proj}_R)$ is inverse to i. $\qquad\square$

Lemma 4.10. *Let R be a connective ring spectrum and let M be a perfect R-module with Tor-amplitude contained in $[0, \infty]$. Then M is connective.*

Proof. This follows from the convergent spectral sequence

$$\mathrm{Tor}_p^{\pi_* R}(\pi_q M, \pi_0 R) \implies \pi_{p+q}(M \wedge_R H\pi_0 R);$$

in particular, if $\pi_q M \neq 0$ for some $q < 0$, then $\pi_q M \otimes_{\pi_* R} \pi_0 R \neq 0$ since R is connective, giving nonzero classes in $\pi_q(M \wedge_R H\pi_0 R)$. $\qquad\square$

Finally, we want to complete the computation by showing that $K_0(\mathrm{End}\,\mathrm{Proj}_R) \cong K_0(\mathrm{End}\,\mathrm{Proj}_{\pi_0 R})$. To do so, we require a technical lemma about rigidifying homotopy commutative triangles in an ∞-category $\mathcal{C}$.

Lemma 4.11. *Let $\mathcal{C}$ be an ∞-category and let K be a 2-skeletal simplicial set. Then any diagram*

$$\sigma\colon K \longrightarrow \mathrm{N}(\mathrm{Ho}(\mathcal{C})),$$

lifts to a diagram $\tau\colon K \to \mathcal{C}$ such that $\eta \circ \tau = \sigma$, where $\eta : \mathcal{C} \to \mathrm{N}(\mathrm{Ho}(\mathcal{C}))$ denotes the unit of the localization $\mathrm{Ho}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}\colon \mathrm{N}$.

Proof. First suppose $K = \Delta^2$ and that $\mathcal{C} = \mathrm{N}(\mathcal{C}')$ for some fibrant simplicial category $\mathcal{C}'$ such that $\mathrm{Ho}(\mathcal{C}') \cong \mathrm{Ho}(\mathcal{C})$. By adjunction, a 2-simplex of $\mathrm{N}(\mathcal{C}')$ is a map $\tau : \mathfrak{C}[\Delta^2] \to \mathcal{C}'$, which is to say objects τ_i, $0 \le i \le 2$, maps $\tau_{ji} : \tau_i \to \tau_j$, $0 \le i < j \le 2$, and a homotopy $\tau_{210} : \tau_{20} \to \tau_{21} \circ \tau_{10}$ in $\mathrm{map}_{\mathcal{C}'}(\tau_0, \tau_2)$. Since we're given a map $\sigma : \mathfrak{C}[\Delta^2] \to \mathrm{Ho}(\mathcal{C}')$, we have homotopy classes of maps $\sigma_{ji} \in \pi_0 \mathrm{map}_{\mathcal{C}}(\sigma_i, \sigma_j)$ such that $\sigma_{20} = \sigma_{21} \circ \sigma_{10}$. Taking $\tau_i = \sigma_i$, we may choose representative $\tau_{ji} : \tau_i \to \tau_j$ of σ_{ji}, and as the two resulting maps τ_{20} and $\tau_{21} \circ \tau_{10}$ from τ_0 to τ_2 are homotopic, we may also choose a 1-simplex $\tau_{210} : \Delta^1 \to \mathrm{map}_{\mathcal{C}'}(\tau_0, \tau_2)$ realizing this.

For the general case, note that there exists a categorical equivalence

$$f : \mathrm{N}(\mathfrak{C}[\mathcal{C}]^{\mathrm{fib}}) \longrightarrow \mathcal{C}$$

which we may suppose is an isomorphism on homotopy categories. First lift the 1-skeleton $\mathrm{sk}_1 K$ to $\mathrm{N}(\mathfrak{C}[\mathcal{C}]^{\mathrm{fib}})$ by choosing representative for homotopy classes of arrows in $\mathcal{C}$, and then extend this to the 2-skeleton by choosing lifts of each 2-simplex compatibly with the chosen lifts on the boundary. Composing with f then gives the desired lift to $\mathcal{C}$. $\qquad\square$

Corollary 4.12. *Let $\mathcal{C}$ be an ∞-category. Then the canonical map*

$$\mathrm{Ho}(\mathrm{End}(\mathcal{C})) \longrightarrow \mathrm{End}(\mathrm{Ho}(\mathcal{C}))$$

is surjective on equivalence classes of arrows.

Proof. Since $\Delta^1 \times \Delta^1/\partial\Delta^1$ is 2-skeletal, the map

$$\pi_0 \mathrm{map}(\Delta^1 \times \Delta^1/\partial\Delta^1, \mathcal{C}) \longrightarrow \pi_0 \mathrm{map}(\Delta^1 \times \Delta^1/\partial\Delta^1, \mathrm{N}(\mathrm{Ho}(\mathcal{C})))$$

is surjective by lemma 4.11. But, by adjunction, the source is isomorphic to the set of equivalence classes of arrows in $\mathrm{Ho}(\mathrm{End}(\mathcal{C}))$, and the target is isomorphic to the set of equivalence classes of arrows in $\mathrm{End}(\mathrm{Ho}(\mathcal{C}))$. $\qquad\square$

Proposition 4.13. *If R is a connective ring spectrum, then the canonical functor*

$$i\colon K_0(\mathrm{End}\,\mathrm{Proj}_R)) \longrightarrow K_0(\mathrm{End}\,\mathrm{Ho}(\mathrm{Proj}_R))$$

is an isomorphism.

Proof. Using the presentation for K_0 of 2.7, we first observe that i is surjective, as

$$\mathrm{End}(\mathrm{Proj}_R)^{\simeq} \longrightarrow \mathrm{N}(\mathrm{End}(\mathrm{Ho}(\mathrm{Proj}_R)))^{\simeq}$$

is surjective on π_0. To see that i is also injective, we must show that any exact sequence in $\mathrm{End}\,\mathrm{Ho}(\mathrm{Proj}_R)$ lifts to an exact sequence in $\mathrm{End}\,\mathrm{Proj}_R$. Since an exact sequence in $\mathrm{End}\,\mathrm{Proj}_R$ is in particular a cofiber sequence, any exact sequence in $S_2^\infty(\mathrm{End}\,\mathrm{Proj}_R)$ is determined (up to contractible ambiguity) by a suitable arrow $\Delta^1 \to \mathrm{End}\,\mathrm{Proj}_R$. Thus, the vertical fibers in the commutative square

$$
\begin{array}{ccc}
S_2^\infty(\mathrm{End}\,\mathrm{Proj}_R)^{\simeq} & \longrightarrow & S_2^\infty(\mathrm{End}\,\mathrm{N}\,\mathrm{Ho}(\mathrm{Proj}_R))^{\simeq} \\
\downarrow & & \downarrow \\
\mathrm{map}(\Delta^1, \mathrm{End}\,\mathrm{Proj}_R) & \longrightarrow & \mathrm{map}(\Delta^1, \mathrm{N}\,\mathrm{Ho}(\mathrm{Proj}_R)),
\end{array}
$$

in which the vertical maps are the restrictions along $\Delta^{\{0,1\}} \to \Delta^2$, are contractible, so the diagram is cartesian. But the bottom horizontal map is surjective on π_0 by corollary 4.12, so the top horizontal map must be surjective on π_0 as well. $\square$

Corollary 4.14. *The map*

$$\operatorname{End} \operatorname{Ho}(\operatorname{Proj}_R) \longrightarrow \operatorname{End} \operatorname{Proj}_{\pi_0 R}$$

induced by the equivalence $\pi_0 \colon \operatorname{Ho}(\operatorname{Proj}_R) \simeq \operatorname{Proj}_{\pi_0 R}$ *is a K_0-isomorphism.*

Proof. This is immediate from proposition 4.13. $\square$

Finally, assembling the comparisons of Corollary 4.14, Proposition 4.13, and Theorem 4.9, we obtain the following:

Theorem 4.15. *For every connective ring spectrum R one has an isomorphism*

$$K_0(\operatorname{End}(\operatorname{Perf}_R)) \cong K_0(\operatorname{End}(\mathbf{P}(\pi_0(R)))) = K_0(\operatorname{End}(\pi_0 R))$$

of abelian groups.

5. Natural operations

In this section we make use of the theory of noncommutative motives to classify all the natural transformations of the above functor (3.6); see Theorem 5.7. The constructions of §3.1 are functorial on exact categories and hence (after the usual fixes associated to the fact that the passage from rings to exact categories of modules is only a pseudo-functor, e.g., see [10, 9.1]) gives rise to a well-defined functor

$$(5.1) \qquad \qquad \operatorname{KEnd}(\mathbf{P}(-)) \colon \operatorname{Rings} \longrightarrow \mathcal{S}$$

from ordinary rings to symmetric spectra. In particular, we have the classical functor

$$(5.2) \qquad \qquad K_0(\operatorname{End}(\mathbf{P}(-))) \colon \operatorname{Rings} \longrightarrow \operatorname{Ab}$$

from ordinary rings to abelian groups. As explained by Almkvist in [1, page 339], a major problem is the computation of all the natural transformations of the above functor (5.2); see also [36][35, §1]. Classical examples are given by the Frobenius operations $F_n \colon [(M, \alpha)] \mapsto [(M, \alpha^n)], n \geq 0$, and by the Verschiebung operations $V_n \colon [(M, \alpha)] \mapsto [(M^{\oplus n}, V_n(\alpha))]$, where

$$(5.3) \qquad V_n(\alpha) := \begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{n+1}\alpha \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(n \times n)}.$$

These natural operations can be generalized to the ∞-categorical setting as follows:

Definition 5.4. (Frobenius) Let f_n be the endofunctor of $\underline{\mathbb{N}}$ induced by the monoid map $n \colon \mathbb{N} \to \mathbb{N}$ which sends m to nm. Using the equivalence

$$\operatorname{Fun}(\operatorname{N}(\underline{\mathbb{N}}), \mathcal{A}) \longrightarrow \operatorname{Fun}(\Delta^1/\partial\Delta^1, \mathcal{A})$$

induced by the categorical equivalence $\Delta^1/\partial\Delta^1 \to \operatorname{N}(\underline{\mathbb{N}})$, one obtains by pre-composition with f_n an exact functor $f_n^* \colon \operatorname{End}(\mathcal{A}) \to \operatorname{End}(\mathcal{A})$ and consequently a map of symmetric spectra $\operatorname{KEnd}(\mathcal{A}) \to \operatorname{KEnd}(\mathcal{A})$. This construction is natural

on $\mathcal{A}$ and hence gives rise to a natural transformation $F_n\colon \mathrm{KEnd} \Rightarrow \mathrm{KEnd}$ of the functor (3.6) that we call the nth-*Frobenius operation.*

For an ∞-category $\mathcal{C} \in \mathrm{Cat}_\infty$, there is a natural functor $\mathcal{C} \to \mathrm{End}(\mathcal{C})$, induced by projection $\Delta^1\partial\Delta^1 \to \Delta^0$, that takes x to $\mathrm{id}\colon x \to x$. By precomposing with the forgetful functor $\mathrm{End}(\mathcal{C}) \to \mathcal{C}$, we obtain the composite functor

$$(5.5) \qquad\qquad \iota\colon \mathrm{End}(\mathcal{C}) \longrightarrow \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{C})$$

which sends the endomorphism $\alpha : x \to x$ to $\mathrm{id} : x \to x$. Moreover, given an ∞-category $\mathcal{D}$ with finite coproducts, given functors $f_i\colon \mathcal{C} \to \mathcal{D}$ we can construct a functor $\coprod f_i\colon \mathcal{C} \to \mathcal{D}$ as the composite

$$\mathcal{C} \longrightarrow \textstyle\prod_i \mathcal{C} \xrightarrow{\ \prod f_i\ } \prod_i \mathcal{D} \longrightarrow \mathcal{D},$$

where the last map is a choice of functorial coproduct. Similarly, given $\tau \in \Sigma_n$, we can permute the factors of this coproduct by τ. We now use these constructions to generalize the Verschiebung:

Definition 5.6. (Verschiebung) For each ∞-category $\mathcal{A} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ there is an endofunctor on $\mathrm{End}(\mathcal{A})$ define by applying the cyclic permutation to the coproduct of $(n-1)$ copies of ι and one copy of $(-1)^{n+1}\,\mathrm{id}$. This functor gives rise to a natural transformation $\mathrm{End}(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$ and hence a natural transformation $V_n\colon \mathrm{KEnd} \to \mathrm{KEnd}$ that we call the nth-*Verschiebung operation.*

Our solution to the problem stated by Almkvist if the following: let

$$W_0(\mathbb{Z}[t]) := \left\{ \frac{1 + p_1(t)r + \cdots + p_i(t)r^i + \cdots + p_n(t)r^n}{1 + q_1(t)r + \cdots + q_j(t)r^j + \cdots + q_m(t)r^m} \,\middle|\, p_i(t), q_j(t) \in \mathbb{Z}[t] \right\}$$

be the multiplicative group of fractions of polynomials in the variable r with coefficients in $\mathbb{Z}[t]$ and constant term 1.

Theorem 5.7. *There is a canonical weak equivalence of spectra between the spectrum* $\mathrm{Nat}(\mathrm{KEnd}, \mathrm{KEnd})$ *of natural transformations of the functor* (3.6) *and the spectrum* $\mathrm{KEnd}(\mathrm{Perf}_{\mathbb{S}[t]})$. *In particular, the abelian group* $\pi_0\,\mathrm{Nat}(\mathrm{KEnd}, \mathrm{KEnd})$ *of natural transformations up to homotopy is isomorphic to*

$$K_0\mathrm{End}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq K_0\mathrm{End}(\mathbf{P}(\pi_0\mathbb{S}[t])) \simeq K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[t])) \simeq \mathbb{Z} \oplus W_0(\mathbb{Z}[t])\,.$$

Moreover, under these isomorphisms, the Frobenius operations F_n *are identified with the elements* $(1, 1+r^n t)$ *and the Verschiebung operations* V_n *with the elements* $(n, 1 + rt^n)$.

Proof. The natural equivalence of spectra $\mathrm{Nat}(\mathrm{KEnd}, \mathrm{KEnd}) \simeq \mathrm{KEnd}(\mathrm{Perf}_{\mathbb{S}[t]})$ follows from Lemma 5.8 below (with E the functor (3.6)). The isomorphisms follow from Theorem 4.15 applied to $R = \mathbb{S}[t]$, from the equality $\pi_0\mathbb{S}[t] = \mathbb{Z}[t]$, and from Almkvist's isomorphism (see [2])

$$K_0\mathrm{End}(\mathbf{P}(A) \xrightarrow{\ \cong\ } K_0(A) \oplus W_0(A) \qquad (M, \alpha)] \mapsto ([M], \det(\mathrm{Id} + \alpha r))$$

applied to $A = \mathbb{Z}[t]$. The identifications of F_n and V_n as the elements in question also follows from the preceding computation and Theorem 3.13; on π_0, the operations of Definitions 5.4 and 5.6 give rise to the classical operations on the K-theory of endomorphisms. Specifically, on passage to K_0, it is clear that the operation of Definition 5.4 takes the the class $[M, \alpha]$ to $[M, \alpha^n]$. Moreover, since for any connective ring spectrum R and compact R-module M, $\mathrm{Map}_R(\vee_n M, \vee_n M) \simeq$

$\prod_n \coprod_n \mathrm{Map}_R(M, M)$, on passage to K_0 the operation of Definition 5.6 gives rise to the matrix specified above in equation (5.3). Now Theorem 3.13 coupled with Almkvist's identification of these operations in terms of the isomorphism (5.8) establishes the desired comparison. $\qquad\square$

Lemma 5.8. *For every additive invariant* $E\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}_\infty$ *there is a natural equivalence of spectra*

$$(5.9) \qquad\qquad \mathrm{Nat}(\mathrm{KEnd}, E) \longrightarrow E(\mathrm{Perf}_{\mathbb{S}[t]}) .$$

Proof. As shown in Theorem 3.10(i), the functor $\mathrm{KEnd}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{S}_\infty$ is an additive invariant. Hence, by equivalence (1.2) one obtains well-defined colimit preserving functors

$$\overline{\mathrm{KEnd}}, \overline{E}\colon \mathcal{M}_{\mathrm{add}} \longrightarrow \mathcal{S}_\infty$$

satisfying $\overline{\mathrm{KEnd}} \circ \mathcal{U}_{\mathrm{add}} \simeq \mathrm{KEnd}$ and $\overline{E} \circ \mathcal{U}_{\mathrm{add}} \simeq E$, as well as a natural equivalence of spectra

$$(5.10) \qquad\qquad \mathrm{Nat}(\mathrm{KEnd}, E) \longrightarrow \mathrm{Nat}(\overline{\mathrm{KEnd}}, \overline{E}) .$$

By Theorem 3.10(ii) the functor $\overline{\mathrm{KEnd}}$ is co-represented in $\mathcal{M}_{\mathrm{add}}$ by the noncommutative motive $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]})$. Hence, the ∞-categorical version of the Yoneda lemma [27, §5.1.3] provides an equivalence of spectra

$$(5.11) \qquad\qquad \mathrm{Nat}(\overline{\mathrm{KEnd}}, \overline{E}) \longrightarrow \overline{E}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}) = E(\mathrm{Perf}_{\mathbb{S}[t]}) .$$

By combining (5.10) with (5.11) we obtain then the above equivalence (5.9). $\qquad\square$

6. (Rational) Witt vectors

Witt vectors were introduced in the thirties by E. Witt [40]. Given a commutative ring A, the *Witt ring* $W(A)$ of A is the abelian group of all power series of the form $1 + a_1 r + a_2 r^2 + \cdots$, with $a_i \in A$, endowed with the multiplication $*$ determined by the equality $(1 - a_1 r) * (1 - a_2 r) = (1 - a_1 a_2 r)$. The *rational* Witt ring $W_0(A)$ of A consists of the elements which are quotients of polynomials; that is, those of the form

$$\left\{ \frac{1 + a_1 r + \cdots + a_i r^i + \cdots + a_m r^m}{1 + b_1 r + \cdots + b_j r^j + \cdots + b_n r^n} \,\middle|\, a_i, b_j \in A \right\} \subset W(A);$$

consult [23] for further details.

Recall from [7, §2.3] that the category $\mathrm{Cat}_\infty^{\mathrm{perf}}$ carries a symmetric monoidal structure in which the tensor product $- \otimes^\vee -$ is characterized by the property that functors out of $\mathcal{A} \otimes^\vee \mathcal{B}$ are in correspondence with functors out of the product $\mathcal{A} \times \mathcal{B}$ which preserve colimits in each variable. The $\otimes^\vee$-unit is the ∞-category $\mathrm{Perf}_{\mathbb{S}}$.

Proposition 6.1. *Let M be a monoid in the ∞-category of spaces. Then the ∞-category $\mathrm{Perf}_{\mathbb{S}[M]}$ of perfect modules for the monoid-ring $\mathbb{S}[M]$ carries a canonical counital, coassociative, and cocommutative coalgebra structure in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.*

Proof. First recall that, if $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category in which the tensor product is the coproduct, then the projection $\mathrm{CAlg}(\mathcal{C}^\otimes) \to \mathcal{C}$ is an equivalence. Taking $\mathcal{C}$ to be the opposite of the ∞-category of A_∞-spaces, we see that M has a coassociative and cocommutative coalgebra structure, so that $\mathbb{S}[M] \simeq \Sigma_+^\infty M$ is a coassociative and cocommutative object in A_∞-spectra. Now, we note that the "one-object spectral category" functor $\mathrm{Alg}_{\mathbb{S}} \to \mathrm{N}(\mathrm{Cat}_{\mathcal{S}})[W^{-1}]$ extends to a

symmetric monoidal $\mathrm{Alg}_{\mathbb{S}}^{\otimes} \to \mathrm{N}(\mathrm{Cat}_{\mathcal{S}})[W^{-1}]^{\otimes}$, and according to [8, §3], $\mathrm{Cat}_{\infty}^{\mathrm{perf}\otimes}$ is a symmetric monoidal localization of $\mathrm{N}(\mathrm{Cat}_{\mathcal{S}})[W^{-1}]^{\otimes}$. Thus $\mathrm{Perf}_{\mathbb{S}[M]}$ inherits a canonical coassociative and cocommutative coalgebra structure. □

Specializing to the case in which $M = \mathbb{N}$, the free A_{∞}-monoid on one generator, this amounts to saying that the diagonal $\Delta \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and the projection $\mathbb{N} \to *$ are monoid maps. Applying Σ_{+}^{∞}, we obtain ring maps

$$\Delta \colon \mathbb{S}[t] \overset{t \mapsto t \wedge t}{\longrightarrow} \mathbb{S}[t] \wedge \mathbb{S}[t] \qquad \epsilon \colon \mathbb{S}[t] \overset{t=1}{\longrightarrow} \mathbb{S}.$$

As proved in [8, §4], the category of noncommutative motives $\mathcal{M}_{\mathrm{add}}$ carries a symmetric monoidal structure making the universal additive invariant $\mathcal{U}_{\mathrm{add}} \colon \mathrm{Cat}_{\infty}^{\mathrm{perf}} \to \mathcal{M}_{\mathrm{add}}$ symmetric monoidal. Hence, by Proposition 6.1, the noncommutative motive $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]})$ becomes a counital coassociative coalgebra in $\mathcal{M}_{\mathrm{add}}$. As a consequence, we obtain the following result:

Proposition 6.2. *The functor*

$$\mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}), -) \colon \mathcal{M}_{\mathrm{add}} \longrightarrow \mathcal{S}$$

is lax symmetric monoidal.

Proof. More generally, we observe that if $\mathcal{C}^{\otimes}$ is a symmetric monoidal stable ∞-category, the bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ specified as the adjoint of the spectral Yoneda embedding lifts to a lax symmetric monoidal bifunctor $(\mathcal{C}^{\mathrm{op}} \times \mathcal{C})^{\otimes} \to \mathcal{S}^{\otimes}$. This follows from the fact that the Yoneda embedding preserves limits and is symmetric monoidal [28, 6.3.1.12] by the relative adjoint functor theorem [28, 7.3.2.7]. Specializing to the case of $\mathcal{M}_{\mathrm{add}}$, this implies in particular that the functor $\mathcal{C} \to \mathcal{S}$ induced by restricting to a commutative algebra in $\mathcal{M}_{\mathrm{add}}^{\mathrm{op}}$ (i.e., a counital coassociative coalgebra) is lax symmetric monoidal. □

Theorem 6.3. *The ring maps $\mathbb{S} \overset{\iota}{\to} \mathbb{S}[t]$ and $\mathbb{S}[t] \overset{t=0}{\to} \mathbb{S}$ give rise to a wedge sum decomposition $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}) \vee \mathbb{W}_0$ of counital coassociative cocommutative coalgebras in $\mathcal{M}_{\mathrm{add}}$. Moreover, for every ordinary commutative ring A, we have an isomorphism of commutative rings*

$$(6.4) \qquad \pi_0 \mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{HA})) \simeq W_0(A).$$

Proof. Recall from Proposition 6.1 that $\mathbb{S}[t] \simeq \mathbb{S}[\mathbb{N}]$ and $\mathbb{S} \simeq \mathbb{S}[*]$ are counital, coassociative, and cocommutative coalgebras. It will be convenient to briefly work with a point-set model of these coalgebras. We can do this using the suspension spectra $\Sigma_{+}^{\infty}\mathbb{N}$ and $\Sigma_{+}^{\infty}*$ in the category of symmetric spectra; these constructions represent coalgebras on the ∞-categorical level because $\Sigma_{+}^{\infty}\mathbb{N}$ is cofibrant. Now, note that the diagrams

$$
\begin{array}{ccc ccc ccc}
\mathbb{S} \wedge \mathbb{S} \overset{\iota \wedge \iota}{\longrightarrow} \mathbb{S}[t] \wedge \mathbb{S}[t] & \quad & \mathbb{S}[t] \wedge \mathbb{S}[t] \overset{(t=0) \wedge (t=0)}{\longrightarrow} \mathbb{S} \wedge \mathbb{S} & \quad & \mathbb{S} = \!\!=\!\! \mathbb{S} \\[4pt]
{\scriptstyle 1 \mapsto 1 \wedge 1} \uparrow \qquad \uparrow {\scriptstyle t \mapsto t \wedge t} & & {\scriptstyle t \mapsto t \wedge t} \uparrow \qquad\qquad \uparrow {\scriptstyle 1 \mapsto 1 \wedge 1} & & \| \qquad \uparrow {\scriptstyle (t=1)} \\[4pt]
\mathbb{S} \overset{\iota}{\longrightarrow} \mathbb{S}[t] & & \mathbb{S}[t] \overset{(t=0)}{\longrightarrow} \mathbb{S} & & \mathbb{S} \overset{\iota}{\to} \mathbb{S}[t]
\end{array}
$$

commute in the category of symmetric spectra. Hence ι and $(t = 0)$ descend to coalgebra maps with ι counit preserving. Since $(t = 0) \circ \iota = \mathrm{Id}$ the composition $\iota \circ (t = 0)$ is an idempotent of $\mathbb{S}[t]$. For the remainder of the proof we work with this idempotent in the ∞-categorical setting.

We can realize this idempotent in terms of the composite maps $* \to \mathbb{N} \to *$ and $\mathbb{N} \to * \to \mathbb{N}$ — these induce a retraction of coalegbras in A_∞ monoids. Applying Σ_+^∞ and the composite functor

$$(6.5) \qquad \mathrm{Cat}_\infty^{\mathrm{ex}} \xrightarrow{(-)^{\mathrm{perf}}} \mathrm{Cat}_\infty^{\mathrm{perf}} \xrightarrow{\mathcal{U}_{\mathrm{add}}} \mathcal{M}_{\mathrm{add}},$$

we obtain then a retraction of coalgebra objects

$$\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}) \longrightarrow \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}) \longrightarrow \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}).$$

Since $\mathcal{M}_{\mathrm{add}}$ is idempotent complete, we may split an idempotent endomorphism $e \colon M \to M$ in $\mathcal{M}_{\mathrm{add}}$ by taking the filtered colimit

$$M_0 \simeq \mathrm{colim}\{M \xrightarrow{e} M \xrightarrow{e} M \xrightarrow{e} \cdots\};$$

taking the fiber $M_1 \to M \to M_0$, we obtain a splitting $M \simeq M_0 \vee M_1$ of M, where e restricts to the identity on M_0 and zero on M_1. Lastly, we note that colimits in the ∞-category of coalgebra objects are computed in the underlying ∞-category. Putting all of this together, we see that we can decompose $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]})$ as a coproduct of $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}})$ together with the coassociative and cocommutative coalgebra object $\mathbb{W}_0$ of $\mathcal{M}_{\mathrm{add}}$. To see that $\mathbb{W}_0$ is counital, we observe that by [28, §5.2.3] it suffices to produce a homotopy counit; the existence of such now follows from proposition A.3 (more generally the arguments of the appendix provide a splitting on the level of the homotopy category).

The identification of $\pi_0 \mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{HA})) \simeq W_0(A)$ follows from the same considerations as in the argument for Theorem 5.7. Specifically, we can identify $\pi_0 \mathrm{Map}(\mathrm{Perf}_{\mathbb{S}[t]}, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{HA})) \cong K_0 \mathrm{End}(\mathrm{Perf}_{HA})$ as $K_0(A) \oplus W_0(A)$, and Almkvist's results [2, pages 2-3] imply that the maps that split off $\mathbb{W}_0$ in the preceding argument split off the $W_0(A)$ component on π_0. This splitting induces the stated commutative ring isomorphism since it is induced by the splitting $\mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}[t]}) \simeq \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_{\mathbb{S}}) \vee \mathbb{W}_0$ of counital coassociative cocommutative coalgebras. $\qquad \square$

Isomorphism (6.4) motivates the following definition:

Definition 6.6. The spectrum of rational Witt vectors of a ∞-category $\mathcal{A} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ is defined as $\mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathcal{A}))$.

The argument for Proposition 6.2 and the fact that $\mathcal{U}_{\mathrm{add}}$ is monoidal [8, §4] yields the following corollary:

Corollary 6.7. *The mapping spectrum* $\mathrm{Map}(\mathbb{W}_0, -)$ *provides a lax symmetric monoidal functor* $\mathcal{M}_{\mathrm{add}} \to \mathcal{S}$. *Therefore, when $\mathcal{A}$ is an E_n object in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, the spectrum of rational Witt vectors* $\mathrm{Map}(\mathbb{W}_0, \mathcal{A})$ *is an E_n object in symmetric spectra.*

Specializing to the case of E_n ring spectra, we have the following further corollary.

Corollary 6.8. *Let R be an E_n ring spectrum. Then the associated rational Witt ring spectrum* $\mathrm{Map}(\mathbb{W}_0, \mathcal{U}_{\mathrm{add}}(\mathrm{Perf}_R))$ *is an E_{n-1} ring spectrum.*

Proof. By Lurie's resolution of Mandell's conjecture (see [28, 8.1.2.6]), the ∞-category of modules for an E_n ring spectrum R is an E_{n-1} object in the ∞-category $\mathrm{Pr}_{\mathrm{St}}^{\mathrm{L}}$ of presentable stable ∞-categories. Since the category of R-modules is compactly generated and the symmetric monoidal structure on the ∞-category $\mathrm{Pr}_{\mathrm{St},\omega}^{\mathrm{L}}$ of compactly-generated stable ∞-categories is induced by the symmetric monoidal

structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{St}}$, we can conclude from [27, §5.5.7] that the ∞-category of perfect modules for an E_n ring spectrum is an E_{n-1} object in $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$. The result now follows from Corollary 6.7. $\qquad\square$

Appendix A. Splitting coalgebras

In this appendix we verify some technical results about splitting of (point-set) coalgebras. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. Recall that a *coalgebra* (A, μ_A, η_A) in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$ and two maps $\mu_A \colon A \to A \otimes A$ (the comultiplication) and $\eta_A \colon A \to \mathbf{1}$ (the counit). If the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\mu_A} A \otimes A \xrightarrow{\mu_A \otimes \mathrm{Id}} & (A \otimes A) \otimes A \\
\downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \simeq} \\
A \otimes A & \xrightarrow{\mathrm{Id} \otimes \mu_A} & A \otimes (A \otimes A)
\end{array}
$$

commutes we say that (A, μ_A, η_A) is *co-associative*, and if the diagram

(A.1)
$$
\begin{array}{ccc}
A & \xleftarrow{\eta_A \otimes Id} \quad A \otimes A \quad \xrightarrow{\mathrm{Id} \otimes \eta_A} & A \\
& \uparrow{\scriptstyle \mu_A} & \\
& A &
\end{array}
$$

commutes we say that (A, μ_A, η_A) is *counital*. Finally, (A, μ_A, η_A) is *cocommutative* if in addition the diagram

(A.2)
$$
\begin{array}{ccc}
A & \xrightarrow{\mu_A} & A \otimes A \\
\downarrow{\scriptstyle \mu_A} & \nearrow{\scriptstyle \tau_{A,A}}^{\simeq} & \\
A \otimes A & &
\end{array} ,
$$

commutes, where $\tau_{A,A}$ stands for the symmetry constraint. A *coalgebra map* $f \colon (A, \mu_A, \eta_A) \to (B, \mu_B, \eta_B)$ consists of a map $f \colon A \to B$ in $\mathcal{C}$ making the diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
\uparrow{\scriptstyle \mu_A} & & \uparrow{\scriptstyle \mu_B} \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes. When $\eta_B \circ f = \eta_A$ we say that it is *counit preserving*.

Now, let us assume that the symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is moreover additive. Note that given counital coassociative coalgebras (A, μ_A, η_A) and (B, μ_B, η_B), the direct sum $A \oplus B \in \mathcal{C}$ becomes a counital coassociative coalgebra. Its comultiplication is given by

$$
\mu_{A \oplus B} \colon A \oplus B \xrightarrow{\mu_A \oplus \mu_B} (A \otimes A) \otimes (B \otimes B) \subset (A \oplus B) \otimes (A \oplus B)
$$

and its counit is given by

$$
\eta_{A \oplus B} \colon A \oplus B \xrightarrow{\eta_A \oplus \eta_B} \mathbf{1} \oplus \mathbf{1} \xrightarrow{\nabla} \mathbf{1} .
$$

Moreover, if (A, μ_A, η_A) and (B, μ_B, η_B) are cocommutative the same holds for the direct sum $A \oplus B$. We now have all the ingredients needed for the following result:

Proposition A.3. *Consider the following diagram in* $\mathcal{C}$

$$ C \underset{f}{\overset{r}{\rightleftarrows}} A \underset{g}{\overset{s}{\rightleftarrows}} B \,. $$

Assume that $r \circ f = \mathrm{Id}$, $g \circ s = \mathrm{Id}$ *and* $f \circ r + s \circ g = \mathrm{Id}$. *Assume also that* A *and* B *are counital coassociative coalgebras and that* s *and* g *are coalgebra maps with* s *counit preserving. Under these assumptions,* C *becomes a counital coassociative coalgebra and* r *and* f *coalgebra maps with* f *counit preserving. Moreover, if* A *and* B *are cocommutative the same holds for* C. *Furthermore, the induced isomorphism* $[r \; g] \colon A \overset{\sim}{\to} C \oplus B$ *is a counit preserving coalgebra map.*

Proof. Let us start by constructing the comultiplication on C. Note that it follows from our hypothesis that C identifies with the cokernel of s. Hence, consider the following commutative diagram

$$
\begin{array}{ccccccc}
B \otimes B & \overset{s \otimes s}{\longrightarrow} & A \otimes A & \longrightarrow & \mathrm{cok}(s \otimes s) & \longrightarrow & \mathrm{cok}(s) \otimes \mathrm{cok}(s) \\
{\scriptstyle \mu_B}\big\uparrow & & {\scriptstyle \mu_A}\big\uparrow & & \big\uparrow & & \big\uparrow {\scriptstyle =: \mu_C} \\
B & \underset{s}{\longrightarrow} & A & \underset{r}{\longrightarrow} & \mathrm{cok}(s) & = & \mathrm{cok}(s) \,.
\end{array}
$$

The left-hand-side square commutes since s is a coalgebra map; the middle one commutes since the dashed arrow is induced by the universal property of the cokernel; and the right-hand-side one commutes since the composition

$$ B \oplus B \overset{s \otimes s}{\longrightarrow} A \otimes A \overset{r \otimes r}{\longrightarrow} \mathrm{cok}(s) \otimes \mathrm{cok}(s) $$

is trivial. The comultiplication μ_C on C is then given by the vertical arrow on the right-hand-side. With this definition it is clear that r becomes a coalgebra map. Note also that since μ_A and μ_B are co-associative the same holds for μ_C. Similarly, if by hypothesis μ_A and μ_B are cocommutative the same holds for μ_C.

Let us now prove that f is also a coalgebra map. Consider the diagram:

$$
\text{(A.4)} \qquad
\begin{array}{ccccc}
C \otimes C & \overset{f \otimes f}{\longrightarrow} & A \otimes A & \overset{g \otimes g}{\longrightarrow} & B \otimes B \\
{\scriptstyle \mu_C}\big\uparrow & & {\scriptstyle \mu_A}\big\uparrow & & \big\uparrow {\scriptstyle \mu_B} \\
C & \underset{f}{\longrightarrow} & A & \underset{g}{\longrightarrow} & B \,.
\end{array}
$$

One needs to show that the left-hand-side square is commutative. By hypothesis g is a coalgebra map and so the right-hand-side square is commutative. Moreover, since $g \circ f = 0$, the outer square is also commutative (since both maps from C to $B \otimes B$ are trivial). This implies that the two maps

$$ C \overset{\mu_C}{\longrightarrow} C \otimes C \overset{f \otimes f}{\longrightarrow} A \otimes A \overset{g \otimes g}{\longrightarrow} B \otimes B \qquad C \overset{f}{\longrightarrow} A \overset{\mu_A}{\longrightarrow} A \otimes A \overset{g \otimes g}{\longrightarrow} B \otimes B $$

agree. Since $g \otimes g$ is surjective (note that it admits a section $s \otimes s$) we then conclude that $(f \otimes f) \circ \mu_C = \mu_A \circ f$, i.e., that the above left-hand-side square in (A.4) commutes.

Let us now define the counit of C as the composition $\mu_C \colon C \xrightarrow{f} A \xrightarrow{\eta_A} \mathbf{1}$. Note that proving the commutativity of diagram (A.1) amounts to show that both composites

$$C \xrightarrow{\mu_C} C \otimes C \xrightarrow{\mathrm{Id} \otimes f} C \otimes A \xrightarrow{\mathrm{Id} \otimes \eta_A} C \qquad C \xrightarrow{\mu_C} C \otimes C \xrightarrow{f \otimes \mathrm{Id}} A \otimes C \xrightarrow{\eta_A \otimes \mathrm{Id}} C$$

are the identity. The proof is similar and so we restrict ourselves to the left-hand-side case. Consider the following commutative diagram

$$
\begin{array}{ccccccc}
C & \xrightarrow{\mu_C} & C \otimes C & \xrightarrow{\mathrm{Id} \otimes f} & C \otimes A & \xrightarrow{\mathrm{Id} \otimes \eta_A} & C \\
\uparrow{\scriptstyle r} & & \uparrow{\scriptstyle r \otimes r} & & & & \| \\
A & \xrightarrow{\mu_A} & A \otimes A & \xrightarrow{\quad r \otimes (\eta_A \circ f \circ r) \quad} & & & C \\
\uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f \otimes f} & & & & \\
C & \xrightarrow{\mu_C} & C \otimes C & & & &
\end{array}
$$

Since $r \circ f = \mathrm{Id}$ it suffices to show that the composite

(A.5) $$C \xrightarrow{f} A \xrightarrow{\mu_A} A \otimes A \xrightarrow{r \otimes (\eta_A \circ f \circ r)} C$$

is the identity map. Since $f \circ r = (\mathrm{Id} - s \circ g)$, we have $r \otimes (\eta_A \circ f \circ r) = r \otimes \eta_A - r \otimes (\eta_A \circ s \circ g)$ and hence, since $(r \otimes \eta_A) \circ \mu_A = r$, we obtain the equality

$$(r \otimes (\eta_A \circ f \circ r)) \circ \mu_A = r - (r \otimes (\eta_A \circ s \circ g)) \circ \mu_A \,.$$

Now, note that $g \circ f = 0$. Since s is a monomorphism it suffices to show that $s \circ g \circ f = 0$ which follows from the equalities:

$$s \circ g \circ f = s \circ g \circ (f \circ r \circ f) = (s \circ g) \circ (\mathrm{Id} - s \circ g) \circ f = (s \circ g - s \circ g) \circ f = 0 \,.$$

The equalities $g \circ f = 0$ and $\mu_A \circ f = (f \otimes f) \circ \mu_C$ allows us then to conclude that

$$(r \otimes (\eta_A \circ f \circ r)) \circ \mu_A \circ f = r \circ f = \mathrm{Id} \ .$$

This shows that C is also a counital coalgebra and that f is counit preserving.

Let us finally prove that the induced isomorphism $[r\ \ g] \colon A \xrightarrow{\sim} C \oplus B$ is a counit preserving coalgebra map. The commutative squares

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{r \otimes r} & C \otimes C \\
\uparrow{\scriptstyle \mu_A} & & \uparrow{\scriptstyle \mu_C} \\
A & \xrightarrow{\quad r \quad} & C
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes A & \xrightarrow{g \otimes g} & B \otimes B \\
\uparrow{\scriptstyle \mu_A} & & \uparrow{\scriptstyle \mu_B} \\
A & \xrightarrow{\quad g \quad} & B
\end{array}
$$

imply automatically that $[r\ \ g]$ is a coalgebra map. In what concerns the counit, one needs to prove that the following diagram commutes

$$
\begin{array}{ccc}
A & \xrightarrow{[r\ \ g]} & C \oplus B \\
\downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_C \oplus \eta_B} \\
& & \mathbf{1} \oplus \mathbf{1} \\
& & \downarrow{\scriptstyle \nabla} \\
\mathbf{1} & =\!=\!=\!= & \mathbf{1} \,.
\end{array}
$$

Note first that since by hypothesis s is counit preserving we have $\eta_B \circ g = \eta_A \circ s \circ g$. On the other hand, by the above definition of η_C we have $\eta_C \circ r = \eta_A \circ f \circ r$. As a consequence, we obtain the equality

$$\eta_C \circ r + \eta_B \circ g = \eta_A(f \circ r + s \circ g) = \eta_A$$

and thus conclude that the above diagram commutes. This concludes the proof. $\square$

REFERENCES

[1] G. Almkvist. *K*-theory of endomorphisms. J. Algebra **55** (1978), 308–340.

[2] ———. The Grothendieck ring of the category of endomorphisms. J. Algebra **28** (1974), 365–388.

[3] B. Antieau and D. Gepner. Brauer groups and étale cohomology of commutative ring spectra. Preprint, arXiv:1210.0290.

[4] C. Barwick, On the algebraic K-theory of higher categories. Preprint, arXiv:1204.3607.

[5] ———, On exact ∞-categories and the Theorem of the Heart. Preprint, arXiv:1212.5232.

[6] S. Betley and C. Schlichtkrull. The cyclotomic trace and curves on *K*-theory. K-Theory **36** (2005), 291-303.

[7] A. J. Blumberg, D. Gepner, and G. Tabuada, A universal characterization of higher algebraic *K*-theory. Available at arXiv:1001.2282v4. To appear in Geometry and Topology.

[8] ———, Uniqueness of the multiplicative cyclotomic trace. Preprint, arXiv:1103.3923v2.

[9] S. Bloch. Algebraic *K*-theory and crystalline cohomology. Pub. math. IHES **47** (1977), 187–268.

[10] A. J. Blumberg and M. A. Mandell. Algebraic *K*-theory and abstract homotopy theory. Adv. in Math., **226** (2011), 3760-3812

[11] A. J. Blumberg and M. Mandell. Localization theorems in topological Hochschild homology and topological cyclic homology. Geom. and Top. **16** (2012), 1053-1120.

[12] W.G. Dwyer and D.M. Kan. Simplicial localizations of categories. J. of Pure and Appl. Alg., **17** (1980), 267–284.

[13] ———. Calculating simplicial localizations. J. of Pure and Appl. Alg., **18** (1980), 17–35.

[14] W. G. Dwyer and D. M. Kan. Function complexes for diagrams of simplicial sets. Indag. Math. **45** (1983), 139-147.

[15] W. G. Dwyer and P. S. Hirschhorn and D. M. Kan and J .H. Smith. Homotopy limit functors on model categories and homotopical categories. AMS Mathematical Surveys and Monographs, **113**, (2004).

[16] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[17] T. Fiore and W. Lueck. Waldhausen additivity: Classical and quasicategorical. Preprint, arXiv:1207.6613.

[18] D. Grayson. K_2 and the *K*-theory of automorphisms. J. Algebra **58** (1979), 12–30.

[19] D. Grayson. The *K*-theory of endomorphisms. J. Algebra **48** (1977), 439–446 .

[20] D. Grayson. Grothendieck rings and Witt vectors. Comm. Algebra **6** (1978), 249–255.

[21] A. Grothendieck et. al., SGA 6.

[22] M. Hazewinkel. Operations in the *K*-theory of endomorphisms. J. Algebra **84** (1983), 283–304.

[23] M. Hazewinkel, *Witt vectors I*. Handbook of algebra. Vol. **6** (2009), 319–472.

[24] L. Hesselholt and I. Madsen. Cyclic polytopes and the K-theory of truncated polynomial algebras. Invent. Math. **130** (1997), 73–97.

[25] L. Hesselholt. On the p-typical curves in Quillen's *K*-theory. Acta Math. **177** (1996), 1–53.

[26] M. Hovey, *Model categories*. Mathematical Surveys and Monographs **63**. American Mathematical Society, Providence, RI, 1999.

[27] J. Lurie, *Higher topos theory*. Annals of Mathematics Studies **170**, Princeton University Press, 2009.

[28] ———, *Higher algebra*. Available at `http://www.math.harvard.edu/ lurie/`.

[29] A. Lindenstrauss and R. McCarthy. On the algebraic *K*-theory of formal power series. J. of K-theory, **10** (2012), 165–189.

[30] M. A. Mandell and B. Shipley. A telescope comparison lemma for THH. Topology and Appl. **117** (2002), no. 2, 161–174.

[31] A. Neeman, *Triangulated categories.*

[32] D. Quillen, *Higher algebraic K-theory I.* Lecture notes in Mathematics **341** (1973), 85–147.

[33] S. Schwede, *Symmetric spectra.* Book project, `http://www.math.uni-bonn.de/ schwede/`.

[34] S. Schwede and B. Shipley, *Stable model categories are categories of modules.* Topology **42** (2003), no. 1, 103–153.

[35] J. Stienstra, *Cartier-Dieudonné theory for Chow groups.* J. Reine Angew. Math. **355** (1985), 1–66.

[36] J. Stienstra, *Operations in the higher K-theory of endomorphisms.* Current trends in algebraic topology, Part 1 (London, Ont., 1981), pp. 59–115, CMS Conf. Proc., 2, Amer. Math. Soc., Providence, R.I., 1982.

[37] G. Tabuada, *Homotopy theory of spectral categories.* Adv. in Math. **221**(4) (2009), 1122–1143.

[38] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories.* The Grothendieck Festschrift, vol. III, Progress in Mathematics, **88**, 247–435.

[39] F. Waldhausen, *Algebraic K-theory of spaces.* Algebraic and geometric topology (New Brunswick, N. J., 1983), 318–419, Lecture Notes in Math., **1126**, Springer, Berlin, 1985.

[40] E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern.* J. Reine Angew. Math. **176** (1937), 31–44.

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